WHEN IS A COMPLEX FIBERED BY A SUBCOMPLEX?

BY
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Introduction. Suppose that (B, F) is a pair of spaces of the homotopy type of countable CW-complexes. Problem 10 of Massey's list of 1955 [6] poses the question of when (B, F) is homotopically equivalent (i.e., there is a homotopy commutative diagram

$$\begin{array}{ccc} B & \longrightarrow & F \\ \downarrow g & & \downarrow h \\ B' & \longrightarrow & F' \end{array}$$

such that g and h are homotopy equivalences) to a pair (B', F') such that B' is fibered, in the sense of Serre, by F' with base-space Y, for some suitable Y. If B' is contractible and F' fibres B', then F' (and hence F) is of the homotopy type of the loopspace $\Omega(Y)$ (see e.g. [9]). Conversely, if B is contractible and if F is of the homotopy type of a loopspace $\Omega(Y)$, then there is a pair (B', F') homotopically equivalent to (B, F) such that F' fibres B' with base-space Y (see [1], [11]). In both cases, Y is of homotopy type of the projective space of F, which is inductively defined by means of Hopf fibrations. Since projective spaces and Hopf fibrations have proved useful in the study of H-spaces, I have tried to preserve these features in the solution of the general problem which I offer in this paper.

The general idea of the solution is to consider first $\Omega(B, F)$, the space of paths beginning in some fixed point b^0 and ending in F, as a right $\Omega(B)$-space, and then to investigate certain multiplications on $\Omega(B, F)$ which extend this action. These multiplications yield fibrations of the Hopf type, in which the total spaces are the manifold joins of F with itself when F is a suspension and are "twisted" manifold joins of F in general (see §6). One also gets spaces which resemble the classical projective spaces; in particular, one gets a filtration of Y (or rather a space of the homotopy type of Y) by subspaces determined by F and B (see §6).

The solution is given in the context of constructions of the reduced product type (RPT-constructions for short). To be able to handle the right action of $\Omega(B)$ on $\Omega(B, F)$, one needs a relative theory of RPT-constructions. This can easily be

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developed as follows. (All complexes are supposed to be special, i.e., countable CW-complexes with a single 0-cell.) Suppose A is an H-complex which is a limit of RPT-complexes [3]. Then A, like RPT-complexes, is essentially an associative H-complex with a nondegenerate multiplication in the sense that the product of two cells of A is again a cell of A. The domain of the theory is the category whose objects are complexes containing A as a subcomplex and on which A acts on both sides nondegenerately in the sense given above. (The objects are called *two-sided A-complexes.*) The maps of the category are cellular maps which respect the action of A. (The maps are called *two-sided invariant maps.*) Having established the category, one proceeds in a manner similar to that of the absolute theory as developed in [3]. Thus if X is a two sided A-complex, then a *set of A-relations in k variables* on X,

$$\mathscr{M}_k(X) = (M_0(X), M_1(X), \cdots, M_k(X); \mu_{p,q}, A),$$

is given by a two-sided A-complex $M_k(X)$ filtered by two-sided A-subcomplexes $M_p(X)$, such that $M_0(X) = A$ and $M_1(X) = X$, and by two-sided invariant maps $\mu_{p,q}$ which are associative, with a two-sided identity, and take $M_p(X) \times M_q(X)$ into $M_{p+q}(X)$ nondegenerately. (If A consists of a single 0-cell only, then this notion of a set of relations is that given in [3] for the absolute case.) One also shows, as in the absolute theory, that X generates a universal H-complex modulo any $\mathscr{M}_k(X)$. The details of the theory are given in the first and third sections.

The application of the relative theory of RPT-constructions to the problem mentioned at the beginning of this introduction is a continuation of the reformulation of Stasheff's A_n-structures given in [3]. First, replace $\Omega(B)$ by a complex A which is a limit of RPT-complexes and $\Omega(B,F)$ by a right A-complex U. It turns out that the existence of a fibration

$$F' \to B' \to Y$$

such that (B',F') is equivalent to the given pair (B,F) implies, first, that $U \sim \Omega(B,F)$ is of the homotopy type of a two-sided A-complex X and, second, that X admits a set of A-relations for all $n \geqq 1$

$$\mathscr{M}_n(X) = (M_0(X), M_1(X), \cdots, M_n(X); \mu_{p,q}, A)$$

such that M_i is a deformation retract of M_{i+1}, where $1 \leqq i \leqq n-1$, the deformation being given by two-sided invariant maps. (X is said to admit a *Q_n-structure* if it admits a set of A-relations such as $\mathscr{M}_n(X)$; if A consists of one 0-cell only, then Q_n- and A_n-structures agree (see [3]). Conversely, the existence of a two-sided A-complex X of the homotopy type of $U \sim \Omega(B,F)$ and of a Q_n-structure on X imply together the existence of a fibration,

$$F_n \to B' \to Y_n,$$

such that the pair (B', F_n) is equivalent to the pair (B, F) in a range that increases with n (see §5). This reduces Massey's problem 10 to a problem on the existence of Q_n-structures.

To get the fibrations of the Hopf type from the set of relations defining a Q_n-structure on $X \sim U \sim \Omega(B, F)$ demands a point of view quite different from the one usually employed. This point of view can be roughly described as follows. Suppose G is an associative H-space and consider the kth stage Hopf-fibration of G,

$$\text{(I)} \qquad G \xrightarrow{i_k} E_k \xrightarrow{p_k} P_k,$$

where E_k is the $(k+1)$-fold join of G with itself and $P_k = E_k/G$, the so-called projective space of G (see [1], [11]). Then p_k induces a fibration

$$\text{(II)} \qquad \Omega P_k \longrightarrow \tilde{G} \xrightarrow{\tilde{i}_k} E_k,$$

from the path-space fibration over P_k,

$$\Omega P_k \longrightarrow P(P_k) \longrightarrow P_k.$$

Note that $\tilde{G}$ is an $\Omega(P_k)$-space of the homotopy type of G, $\tilde{i}_k$ equivalent to i_k, and p_k the classifying map of the fibration (II). In Massey's problem, $M_1(X) = X$ is analagous to G, and ${}^kM$, to ΩP_k where ${}^kM$ is the RPT-complex generated by X modulo $\mathscr{M}_k(X)$. What I do here is construct (II) first and then obtain (I) from it by constructing the classifying map p_k. To construct (II) one must replace M_1 by a homotopically equivalent right ${}^kM$-complex N_k and then show how in general a right A-complex N is quasi-fibered by A over a base-complex $N//A$ similar to an orbit space. The latter problem is answered by Theorem (2.2) of §2—a result of independent interest which plays a primary role in this work. The complex N_k is constructed in §8. Now, Theorem (2.2) shows that N_k is quasi-fibered by A over the orbit complex $N_k//A$. The complex $N_k//A$ is shown to be of the homotopy type of the $(k+1)$-fold join of F with itself, when F is a suspension, and is the "relative" $(k+1)$-fold join of $M_1//A$ in general (see §§6 and 8). The quasi-fibration of any right A-complex N over $N//A$ is shown, in §4, to admit a classifying map. One obtains (I) from (II) by taking p_k to be the classifying maps of (II).

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1. **Preliminary notions and definitions.** Suppose A is a special complex in the sense of James, i.e., a countable CW-complex with a single 0-cell a^0. Take a^0 to be the basepoint of A. Assume that A is an H-complex with multiplication α. (Recall that, by definition, α is cellular, associative, and has a^0 as an identity.) Suppose also that A is filtered by subcomplexes

$$A_0 = \{a^0\} \subset A_1 \subset \cdots \subset A_p \subset \cdots$$

and α takes $A_p \times A_q$ into A_{p+q} for all p and q. Write $A_n(X)$ for A_n, where $A_1 = X$. Then we say that A is a *complex of the reduced product type* (an *RPT-complex* for short) if, and only if, the set

$$\mathscr{A}_k(X) = (A_0(X), A_1(X), \cdots, A_k(X); \alpha_{p,q}),$$

where $\alpha_{p,q} = \alpha \,|\, A_p(X) \times A_q(X)$, is a set of relations in the sense of [3], for all $k \geqq 2$. If, moreover, there is an integer m such that

$$A_p(X) = \bigcup_{r+s=p, r,s>0} \alpha_{r,s}(A_r(X) \times A_s(X)),$$

for all $p > m$, then we say that A is of *rank* $\leqq m$. Otherwise we say A is of *infinite rank.* (This notion of an RPT-complex is slightly different from that given in [3]. An RPT-complex in the sense of [3] is always of finite rank. Here we are going to admit RPT-complexes of arbitrary rank.)

Suppose that B is a special complex containing A as a subcomplex and that it is filtered by an ascending sequence of subcomplexes

$$B_0 = A \subset \cdots \subset B_m \subset \cdots.$$

Assume that there is a cellular mapping

$$\eta: B \times A \to B, \tag{1.1}$$

with the following properties:

$(1.1)_a$ *the basepoint a^0 of A is the identity,* i.e. $\eta(b, a^0) = b$, *for all b in B;*

$(1.1)_b$ *η is associative,* i.e. $\eta(\eta \times 1) = \eta(1 \times \eta)$; *and*

$(1.1)_c$ *η restricted to $A \times A$ is just α, the multiplication in A.*

So far, what we have is a complex B and an action on the right by A on B. However, one must restrict one's attention, in the context of reduced products, to those actions which are nondegenerate in the sense that the product of a cell in B by a cell in A is a cell in B. The precise condition is the following:

$(1.1)_d$ *η takes $B_m \times A_n$ to B_{m+n} and imbeds the complex $[B_m, A_n]$ obtained by attaching $B_m \times A_n$ to B_{m+n-1} by the map $\eta \,|\, B_{m-1} \times A_n \cup B_m \times A_{n-1}$, into B_{m+n} as a subcomplex.*

Definition (1.2). *Suppose B is a special complex and A an RPT-complex imbedded in B as a subcomplex. B is said to be a right A-complex if it is filtered by an ascending sequence*

$$B_0 = A \subset \cdots \subset B_m \subset \cdots$$

and there is a cellular mapping

$$\eta: B \times A \to B$$

subject to the conditions $(1.1)_a, \cdots, (1.1)_d$ given above.

Similarly, we define a *left A-complex B* to be a filtered special complex with a mapping

$$\phi: A \times B \to B$$

subject to conditions similar to $(1.1)_a, \cdots, (1.1)_d$ given above. *A two-sided A-complex* B is, by definition, a left and right A-complex such that the left and right actions of A on B commute with each other.

In studying A-complexes one can vary the given filtration to a certain extent (see §3 below). In fact, one can always replace an arbitrary filtration by the one given by its skeletons, as can be easily checked. Note also that nondegeneracy is defined by means of the filtration, rather than by requiring that the product on any two cells be also a cell, because quite often one gets naturally defined filtrations (see §5), with which it is more natural to work. Note also that the filtrations of the left and right actions a two-sided complex need not be the same.

An example of a two-sided A-complex is given by taking $B = A$ and letting η and ϕ be the multiplication α in A. Other examples are obtained by taking first a special complex C containing A as a subcomplex, and then setting

$$B = A \cup C \times A,$$

where the complex $C \times A$ is attached to A by the multuiplication α of A. Clearly B is a right A-complex with A acting naturally on the right. Similarly, one can define $A \cup A \times C$ and $A \cup A \times C \times A$. The former is a left A-complex, and the latter is a two-sided A-complex. In fact, any A-complex is filtered by subcomplexes of one of these three types. To see this, suppose A is an RPT-complex of arbitrary rank, and let B be a right A-complex. By definition, let $\tilde{B}_1$ be the smallest subcomplex of B which contains A and the cells of $B - A$ of lowest dimension. Then the complex

$$B = A \cup \tilde{B}_1 \times A,$$

where $\tilde{B}_1 \times A$ is attached to A by the multiplication $A \times A \to A$, is a subcomplex of B, as follows easily from $(1.1)_d$. Suppose we have been able to define a sequence of subcomplexes of B,

$$\tilde{B}_0 = A \subset \tilde{B}_1 \subset \cdots \subset \tilde{B}_m,$$

and a sequence of right A-subcomplexes of B,

$$B_0 = A \subset B_1 \subset \cdots \subset B_m,$$

such that

(i) $\tilde{B}_i \supset B_{i-1}$ and

(ii) $B_i = B_{i-1} \cup \tilde{B}_i \times A$,

where the attaching map is given by the action of $B_{i-1} \times A \to B_{i-1}$. If $B_m \neq B$, define $\tilde{B}_{m+1}$ to be the smallest subcomplex of B containing B_m and the cells of $B - B_m$ of smallest dimension. Now, if we let

$$B_{m+1} = B_m \cup B_{m+1} \times A,$$

where the attaching map is given by the action of $B_m \times A \to B_m$, we get what we wish. Hence we have the following proposition.

PROPOSITION (1.3). *Suppose B is a right A-complex, where A is an RPT-complex of arbitrary rank. Then there is a sequence of subcomplexes of B,*

$$A = \tilde{B}_0 \subset \tilde{B}_1 \subset \cdots \subset \tilde{B}_m \subset \cdots,$$

and a sequence of right A-subcomplexes of B,

$$A = B_0 \subset B_1 \subset \cdots \subset B_m \subset \cdots,$$

such that

(i) $B_{m-1} \subset \tilde{B}_m$;
(ii) $B_m = B_{m-1} \cup \tilde{B}_m \times A$, $m \geqq 1$
(the attaching map being the action $B_{m-1} \times A \to B_{m-1}$); and
(iii) $B = \lim_m B_m$.

Similarly, one can prove the following proposition.

PROPOSITION (1.4). *Suppose B is a two-sided A complex, where A is an RPT-complex of arbitrary rank. Then there is an ascending sequence of subcomplexes of B,*

$$A = \tilde{B}_0 \subset \tilde{B}_1 \subset \cdots \subset \tilde{B}_m \subset \cdots,$$

and an ascending sequence of two sided A-subcomplexes of B,

$$A = B_0 \subset B_1 \subset \cdots \subset B_m \subset \cdots,$$

such that

(i) $B_{m-1} \subset \tilde{B}_m$;
(ii) $B_m = B_{m-1} \cup A \times \tilde{B}_m \times A$, $m \geqq 1$
(the attaching map being the action $A \times B_{m-1} \times A \to B_{m-1}$); and
(iii) $B = \lim_m B_m$.

There is a similar representation for the left A-complexes. Note that these representations depend on the cellular decomposition of B and not on the filtrations.

Suppose B is a right A-complex and B' a right A'-complex. By definition, a *right invariant map*

$$f: B \to B'$$

is a cellular map of B into B' such that

$$f\phi(x, a) = \phi'(f(x), f(y)),$$

where ϕ and ϕ' are the actions of A and A' on B and B', for all x in B and a in A, and $f|A$ is a homomorphism of A into A'. Similarly we define a *left invariant* *map*, and a *two-sided invariant map*. A *right* (left, two-sided) *A-map* f is a right (left, two-sided) invariant map such that $A = A'$ and $f|A = 1$.

Suppose next that B and B' are, respectively, right and left A-complexes, and denote the actions of A by ϕ and η'. We wish to join B to B' along the "orbits" of A. Consider, therefore, the special complex

$$B \times A \times B'.$$

Suppose $x_1 = (b_1, a_1, b_1')$ and $x_2 = (b_2, a_2, b_2')$ are two elements of $B \times A \times B'$. Then we shall say that x_1 is an *amalgam* of x_2 if, and only if,

$$b_1 = \phi(b_2, a_2),\ a_1 = a^0, \text{ and } b_1' = b_2', \text{ or}$$

$$b_1' = \eta'(a_2, b_2'),\ a_1 = a^0, \text{ and } b_1 = b_2,$$

where a_1, a_2 are in A, b_1, b_2 in B, and b_1', b_2' in B', and a^0 is the basepoint of A. This notion of amalgamation is similar to that of [3] and generates in a similar manner an equivalence relation. Let $B \times_A B'$ be the quotient space of $B \times A \times B'$ modulo this equivalence relation, and give $B \times_A B'$ the quotient topology. Clearly $B \times_A B'$ is a covariant functor in two variables. The following result can be proved in the same manner as Theorem (2.4) of [3].

THEOREM (1.4). *Suppose B and B' are right and left A-complexes. Then the cellular structure of $B \times A \times B'$ induces the structure of a special complex on $B \times_A B'$. Also, if B and B' are two-sided A-complexes, then the left action of A on B and the right action of A on B' make $B \times_A B'$ a two-sided A-complex.*

By definition, $B \times_A B'$ is the *A-cartesian product* of B and B'. Observe that the imbeddings of B and B' as the first and last factor in $B \times A \times B'$ induce imbeddings of B and B' in $B \times_A B'$, as subcomplexes whose intersection is A. Note also that if $A = \{a^0\}$, a single 0-cell, then an A-complex is just a special complex, and $B \times_A B'$ is the usual cartesian product.

2. **The quasi-fibration of an A-complex over its orbit space.** Recall that a map

$$p: E \to B$$

of the space E onto the space B is a *quasi-fiber map* if, and only if,

$$p_*: \pi_i(E, p^{-1}(x), y) = \pi_i(B, x),$$

for all x in B, y in $p^{-1}(x)$, and $i \geqq 0$ [2]. Also recall that if A is an associative H-space, then a *left principal quasi-fiber map* for A is a quasi-fiber map

$$p: E \to B,$$

together with a left action of A on E,

$$\eta: A \times E \to E,$$

such that

(i) $\eta(a^0, e) = e$,

where a^0 is the 0-cell of A and e is in E;

(ii) $\eta(A \times e) \subset p^{-1}(p(e))$,

$p^{-1}(p(e))$ being the fiber through e; and

(iii) the mapping

$$\eta_e: A \to p^{-1}(p(e))$$

which takes a to $\eta(a, e)$ is a weak homotopy equivalence [1]. Similarly one defines a *right* principal quasi-fibration.

The object of this section is to define the complex of orbits of A in N and to show that the natural projection of N on its orbit complex is a quasi-fibration.

Suppose A is an RPT-complex of arbitrary rank, and let N be a right A-complex as in §1. Denote the action of A on N by η. Two elements n and n' of N are said to be *equivalent* if, and only if, there is a chain of elements $n_1, \cdots, n_r$ in N such that, for every consecutive pair n_s, n_{s+1}, there is an element a in A such that either $\eta(n_s, a) = n_{s+1}$ or $\eta(n_{s+1}, a) = n_s$. Denote the quotient space by $N//A$, and give it the quotient topology. $N//A$ is, by definition, the *orbit complex* of η. Suppose p is the natural projection of N onto $N//A$. Now, according to Proposition (1.3) of §1, there is an ascending sequence

$$N_0 \subset \cdots \subset N_m \subset \cdots \tag{2.1}$$

of right A-subcomplexes of N such that

$$N_0 = A, \text{ and } N_m = N_{m-1} \cup \tilde{N}_m \times A, \qquad m > 0, \tag{2.1a}$$

where $\tilde{N}_m$ is the smallest subcomplex of N containing N_{m-1} and the cells of $N - N_{m-1}$ of lowest dimension. We conclude from this that the cellular structure of N induces the structure of a special complex on $N//A$ and that the natural map,

$$p: N \to N//A,$$

is cellular.

THEOREM (2.2). *Suppose N is a right A-complex. Then the natural projection*

$$p: N \to N//A$$

is a principal quasi-fibration.

Theorem (2.2) is a special case of the following theorem.

THEOREM (2.2′). *Suppose N and N' are right and left A-complexes. Then*

(i) *the map*

$$p' : N \times_A N' \to N // A,$$

induced by the projection on the first factor, is a quasi-fibration with fibre N'; *and*

(ii) *the map*

$$\gamma_n : N' \to p'^{-1}(p(n)),$$

which takes the point n' *to* $n \times_A n'$, *is a homotopy equivalence for all* n *in* N.

Proof. Consider the ascending sequence (2.1) given above. If $m = 0$, then $N_0 // A = \tilde{a}^0$, the 0-cell of $N // A$, and, hence

$$p'_0 : N_0 \times_A N' \to \tilde{a}^0, \qquad p'_0 = p \,|\, N_0$$

is trivially a quasi-fibration; and, since A is arcwise connected, condition (ii) is also satisfied. Suppose we have proved that the map

$$p'_m : N_m \times_A N' \to N_m // A, \qquad p'_m = p \,|\, N_m$$

is a quasi-fibration with fiber N' and such that (ii) of the theorem is satisfied when $N = N_m$. We wish to do the same thing for the case $m + 1$. Note that

$$N_{m+1} - N_m = \bigcup \dot{D}^{n_r} \times A$$

where D^{n_r} ranges over the cells of $\tilde{N}_{m+1} - N_m$ and $\dot{D}^k$ denotes the interior of D^k. Hence,

$$N_{m+1} \times_A N' - N_m \times_A N' = \bigcup \dot{D}^{n_r} \times N'.$$

Moreover, we can write

$$N_{m+1} // A = U \cup V,$$

where

$$V = p'_{m+1} \left(\bigcup D^{n_r}(3/4) \times N' \right)$$

($D^{n_r}(3/4)$ being the n_r-disc of radius 3/4), and

$$U = p'_{m+1}(N_{m+1}(1/4) \times_A N'),$$

where

$$N_{m+1}(1/4) = N_m \cup \bigcup (D^{n_r} - \dot{D}^{n_r}(1/4)) \times A.$$

Clearly V is a distinguished set. Hence, to finish the proof we need to show that U is also distinguished since $U \cap V$ is distinguished (see (2.2) of [2]). But, according to (2.10) of [2], we need prove only that the map

$$\gamma_n : N' \to p'^{-1}_{m+1}(p'_{m+1}(n))$$

which takes n' to $n \times_A n'$, for all n in N_{m+1} is a homotopy equivalence. (This also proves the assertion (ii) of the theorem when $N = N_{m+1}$.) Observe that

$$p_m'^{-1}(p_m'(n)) = p_{m+1}'^{-1}(p_{m+1}'(n)),$$

if n is in N_m. In this case the induction assumption implies γ_n is a homotopy equivalence. If n is in $N_{m+1} - N_m$, then

$$n = (\tilde{n}, a),$$

for some $\tilde{n}$ in $\tilde{N}_{m+1} - N_m$ and a in A. Hence γ_n takes n' to $(\tilde{n}, an')$. Since A is arcwise connected and

$$p_{m+1}'^{-1}(p_{m+1}'(n)) = \tilde{n} \times N',$$

γ_n is a homotopy equivalence. This establishes the induction step and hence the theorem.

REMARK (2.3). Suppose N' is a left A-complex. Then one can define an equivalence relation on N', as was done for right A-complexes. Denote the quotient space by $A \backslash\backslash N'$. Then $A \backslash\backslash N'$ inherits from N' a natural cellular structure, and the projection

$$p' : N' \to A \backslash\backslash N'$$

is again a principal quasi-fibration. Moreover, if N is a right A-complex, then the projection

$$N \times_A N' \to A \backslash\backslash N'$$

induced by the projection on the second factor is a quasi-fibration with fiber N.

3. **Constructions relative to an RPT-complex.** Very often one has a complex X on which an RPT complex A already acts, and one is interested in studying multiplications on X which extend to given action of A. The absolute theory of RPT-constructions given in [3] does not take care of this problem. One needs a relative theory. The object of this section is to develop such a relative theory.

Suppose A is an RPT-complex of arbitrary rank as in §1. Assume that X is two-sided A-complex, and let

$$M_0(X) \subset M_1(X) \subset \cdots \subset M_k(X)$$

be an ascending sequence of two-sided A-complexes such that $M_0(X) = A$, $M_1(X) = X$, and $M_p(X)$ is a subcomplex of $M_{p+1}(X)$ for $k > p \geqq 0$. Suppose also that there is a two-sided invariant map,

(3.1) $$\mu_{p,q} : M_p(X) \times_A M_q(X) \to M_{p+q}(X),$$

for all integers $p, q \geqq 0$ with $p + q \leqq k$, such that the following conditions are satisfied.

$(3.1)_a$ *The map* $\mu_{p,q} | M_p(X) \cup M_q(X)$ *is the folding map;*

$(3.1)_b$ *any two maps* $\mu_{p,q}$ *and* $\mu_{p',q'}$ *agree on their common domain of definition; and*

$(3.1)_c$ *the maps* $\mu_{p,q}$ *are associative, i.e.* $\mu_{p+q,r}(\mu_{p,q} \times 1) = \mu_{p,q+r}(1 \times \mu_{q,r})$ *whenever the mappings are defined.*

Notice that $M_k(X)$ is a two-sided A-complex filtered by two-sided A-subcomplexes and with a partially defined multiplication which is associative and which extends the action of A on both sides, as $(3.1)_a$ easily implies. If $A = \{a^0\}$, then this is precisely the situation described in [3]. Again as in [3], we need another condition on $\mu_{p,q}$ to guarantee that the product of a cell of $M_p(X)$ by a cell of $M_q(X)$ is also a cell of $M_{p+q}(X)$. So, proceeding in a similar fashion, we note that the mappings $\mu_{p,q-1}$ and $\mu_{p-1,q}$ induce together a cellular mapping

$$\delta_{p,q}\colon M_p \times_A M_{q-1} \cup M_{p-1} \times_A M_q \to M_{p+q-1},$$

where M_k stands for $M_i(X)$. By definition,

$$[M_p(X), M_q(X)]_A = M_{p+q-1}(X) \cup \delta_{p,q} M_p(X) \times_A M_q(X)$$

where the complex $M_p(X) \times_A M_q(X)$ is attached to $M_{p+q-1}(X)$ by the mapping $\delta_{p,q}$. It is easy to see that $[M_p(X), M_q(X)]_A$ is a two-sided A-complex and that it contains $M_{p+q-1}(X)$ as a two-sided A-subcomplex. The desired condition is:

$(3.1)_d$ *the map* $\mu_{p,q}$ *induces an imbedding of* $[M_p(X), M_q(X)]_A$ *into* $M_{p+q}(X)$ *as a two-sided A-subcomplex, for all* $p, q \geqq 0$ *and* $p - q \leqq k$.

Definition (3.2). *Suppose X is a two-sided A-complex, and let*

$$A = M_0(X) \subset M_1(X) = X \subset \cdots \subset M_k(X)$$

be an ascending sequence of two-sided A-complexes. Assume that for every pair of integers $p, q \geqq 0$ *such that* $p + q \leqq k$, *we are given a two-sided A-mapping,*

$$\mu_{p,q}\colon M_p(X) \times_A M_q(X) \to M_{p+q}(X),$$

subject to the conditions $(3.1)_a, \cdots, (3.1)_d$ *given above. Then we say that*

$$\mathscr{M}_k(X) = (M_0(X), \cdots, M_k(X); \mu_{p,q}, A)$$

is a set of A-relations in k variables.

Note that if $A = \{a^0\}$, then this notion reduces to that of [3]. Thus any set of relations in the old sense is a set of $\{a^0\}$-relations in the sense of Definition (3.2). One can relativize an absolute set of relations in other ways, and sometimes it is both natural and necessary to do so. In fact, let

$$\mathscr{M}_k(X) = (M_0(X), \cdots, M_k(X), \mu_{p,q})$$

be a set of relations in the sense of [3]. Let $A = X_\infty$, the reduced product complex of James [4]. By definition, let

$$\tilde{M}_p = A \cup_\gamma A \times M'_p \times A,$$

where M'_p is the complex $A \cup M_p(X)$ obtained by attaching $A = X_\infty$ to $M_p(X)$ along the subcomplex X_p and the attaching map

$$\gamma: A \times A \times A \to M'_p$$

is the one induced by the multiplication in A. Observe that $\tilde{M}_0 = A$. Put $\tilde{X} = \tilde{M}_1$ and $M_p(\tilde{X}) = \tilde{M}_p$. Denote by $\tilde{\mu}_{p,q}$ the two-sided invariant maps

$$\tilde{\mu}_{p,q}: M_p(\tilde{X}) \times_A M_q(\tilde{X}) \to M_{p+q}(\tilde{X})$$

induced by $\mu_{p,q}$. Then the set

$$\mathscr{M}_k(\tilde{M}_1) = (M_0(\tilde{X}), \cdots, M_k(\tilde{X}); \tilde{\mu}_{p,q}, A)$$

is a set of A-relations in k variables equivalent to the original set $\mathscr{M}_k(X)$ in the sense that it does not restrict the multiplications $\mu_{p,q}$ any further (see end of section). Clearly one can, in a similar manner, relativize the set $\mathscr{M}_k(X)$ with respect to any of the RPT-complexes ${}^mM$ generated by X modulo the subsets

$$\mathscr{M}_m(X) = (M_0(X), \cdots, M_m(X); \mu_{p,q}), \qquad m \leqq k-1.$$

Under certain circumstances the converse is also true. We shall indicate how it can be done by discussing a simple situation. Suppose $A = X_\infty$, the reduced product complex of a special complex X, and assume that

$$\mathscr{M}_2(\tilde{X}) = (M_0(\tilde{X}), M_1(\tilde{X}), M_2(\tilde{X}); \tilde{\mu}_{p,q}, A)$$

is a set of relations on $\tilde{X}$ in two variables such that

$$\tilde{X} = M_1(\tilde{X}) = A \cup_f A \times D^r \times A,$$

where D^r is the r-disc and the attaching map

$$f: A \times S^{r-1} \times A \to A$$

extends the multiplication in A. Assume also that

$$M_2(\tilde{X}) = [M_1(\tilde{X}), M_1(\tilde{X})]_A.$$

To define an equivalent absolute set of relations, let k be the smallest integer such that $f(S^{r-1}) \subset X_k$, and put

$$M_k(X) = X_k \cup_f D^r.$$

Then the set

$$\mathscr{M}_k(X) = (M_0(X), \cdots, M_k(X); \mu_{p,q}),$$

where $M_0 = x^0$, the 0-cell of X, $M_1 = X$, $M_p = X_p$, for $p \leqq k-1$, M_k is as above, and $\mu_{p,q}$ are the natural maps, is a set of relations in the absolute sense (see [3]) equivalent to the given set $\mathscr{M}_2(\tilde{X})$.

DEFINITION (3.3). *Suppose that*

$$\mathscr{M}_k(X) = (M_0(X), \cdots, N_k(X); \mu_{p,q}, A) \text{ and}$$
$$\mathscr{M}_k(X') = (M_0(X'), \cdots, M_k(X'); \mu'_{p,q}, A')$$

are two sets of two-sided A-and A'-relations on X and X'. A mapping

$$f: \mathscr{M}_k(X) \to \mathscr{M}_k(X')$$

is a tuple of two-sided invariant maps $(f_0, \cdots, f_k)$ *such that*

$$\mu'_{p,q}(f_p \times f_q) = f_{p+q}\mu_{p,q},$$

for all p and q such that $p + q \leqq k$.

Again this concept reduces to that of [3] in the case $A = \{a^0\}$.

Suppose next that M is an H-complex with multiplication μ, and assume that M contains an RPT-complex A of arbitrary rank as a sub H-complex. Suppose also that M is filtered by an ascending sequence of two-sided A-complexes

$$A = M_0 \subset M_1 \subset \cdots \subset M_k \subset \cdots,$$

and assume that μ takes $M_p \times_A M_q$ into M_{p+q}, for all p and $q \geqq 0$. Denote M_1 by X, and write $M_p(X)$ for M_p for all $p \geqq 0$. Then, as in the absolute case (see §1), we say that M is an *RPT-complex relative to A* if, and only if, the set

$$\mathscr{M}_k(X) = (M_0(X), M_1(X), \cdots, M_k(X); \mu_{p,q}, A)$$

where $\mu_{p,q}$ is induced by μ on $M_p(X) \times_A M_q(X)$ is a set of A-relations on X, in the above sense, for all integers $k \geqq 2$. If, moreover, there is an integer m such that

$$\mathscr{M}_p(X) = \bigcup \mu_{r,s}(M_r(X) \times_A M_s(X)), \qquad r, s > 0, \quad r + s = p,$$

for all $p > m$, then we say that M is of *rank* $\leqq m$. Otherwise we say that M is of *infinite rank*. Here also an RPT-complex relative to $\{a^0\}$ is just an RPT-complex in the absolute sense, as defined in the first section.

Next we wish to show that every set of A-relations generates an RPT-complex relative to A. The construction will be seen to be a modification of the one given in [3] for the absolute case.

Suppose that

$$\mathscr{M}_k(X) = (M_0(X), \cdots, M_k(X); \mu_{p,q}, A)$$

is a set of relations as above, where A is an arbitrary RPT-complex. By definition, the index of an element a in A is 0, and the index of an element x in $M_i - M_{i-1}$ is i. If

$$u = (x_1, \cdots, x_n) \in (M_k)^n,$$

where $(M_k)^n$ is the n-fold cartesian product $M_k \times M_k \times \cdots \times M_k$, then the index of u is $\sum_{i=1}^{n}$ index (x_i). Consider now the set

$$S_n = \{u \in (M_k)^n \mid \operatorname{index}(u) \leqq n\}.$$

It can be easily seen that S_n is a two-sided A-subcomplex of $(M_k)^n$. If $u = (x_1, \cdots, x_n)$ and $u' = (x'_1, \cdots, x'_n)$ are in $(M_k)^n$, then u' is said to be an *amalgam*(2) of u if, and only if, there is an integer $p < n$ such that

$$x_q = x'_q, \text{ if } q < p, \text{ or } q > p+1,$$

$$\mu(x_p, x_{p+1}) = x'_p, \text{ and}$$

$$a^0 = x'_{p+1},$$

where a^0 is the 0-cell of A and μ stands for the appropriate multiplication of $\mathscr{M}_k(X)$. The notion of amalgamation generates an equivalence relation on S_n (see [3]). Denote the quotient space by $M'_n(X)$, and give it the quotient topology. Note that A acts on the left and right on $M'_n(X)$ and that the two actions commute with each other. Since the mappings $\mu_{p,q}$ are associative, the juxtaposition map of $S_m \times S_n$ into S_{m+n} induces a continuous mapping

$$(3.4)_{r,s} \qquad \mu'_{r,s}: M'_r(X) \times_A M'_s(X) \to M'_{r+s}(X).$$

Note that the mappings $\mu'_{p,q}$ induce a mapping

$$\delta_{r,s}: M'_r(X) \times_A M'_{s-1}(X) \cup M'_{r-1}(X) \times_A M'_s(X) \to M'_{r+s-1}(X)$$

By definition, let

$$(3.5)_{r,s} \qquad [M'_r(X), M'_s(X)]_A = M'_{r+s-1}(X) \cup_{\delta_{r,s}} M'_r(X) \times_A M'_s(X).$$

Then A acts on both sides of $[M'_r(X), M'_s(X)]_A$, and the actions commute with each other.

THEOREM $(3.6)_{r,s}$. *$M'_n(X)$ inherits naturally from S_n a cellular structure which makes it a two-sided A complex with the following properties*:

$(a)_n$ *there is a natural imbedding β_{n-1} of $M'_{n-1}(X)$ in $M'_n(X)$ as a two-sided A-subcomplex*;

$(b)_n$ *if $r+s=n$, then the map $\mu'_{r,s}$ of $(3.4)_{r,s}$ is cellular, and $[M'_r(X), M'_s(X)]_A$ is a two-sided A-complex imbedded in $M'_n(X)$ as a two-sided A-subcomplex, the imbedding being induced by $\mu'_{r,s}$ of $(3.4)_{r,s}$*;

$(c)_n$ *$M'_n(X)$ is naturally isomorphic to $M_n(X)$ of $\mathscr{M}_k(X)$ as a two-sided A-complex for $n \leqq k$, and $\mu'_{r,s}$ is equal to $\mu_{r,s}$ of $\mathscr{M}_k(X)$ as a two-sided invariant map for $r+s \leqq k$.*

(2) Cf. definition of A-cartesian product in §1.

Note that this theorem is just Theorem $(2.4)_n$ of [3], when $A = \{a^0\}$. The proof is also similar to that case, and therefore will not be given.

From here on write $M_p(X)$ and $\mu_{r,s}$ for $M'_r(X)$ and $\mu'_{r,s}$, and identify $[M_r(X), M_s(X)]_A$ with its image in $M_{r+s}(X)$.

COROLLARY (3.7). *Suppose* $n > k \geqq r \geqq 1$. *Then, if* $r < s$,

$$[M_{n-s}, M_s]_A \cap [M_{n-r}, M_r]_A$$

is isomorphic to $[M_{n-s}, M_{s-r}, M_r]_A$. *Moreover,* M_n *is the union of the two-sided A-complexes* $[M_{n-r}, M_r]_A$, *for* $r = 1, \cdots, k$.

By definition, let

$$M(X) = \lim_n M_n(X).$$

Then $M(X)$ is certainly a two-sided A-complex, and the two-sided invariant mappings $\mu'_{r,s}$ of $(3.4)_{r,s}$ induce a two-sided invariant map

$$\mu : M(X) \times_A M(X) \to M(X).$$

THEOREM (3.8). *$M(X)$ with the multiplication μ is an RPT-complex relative to A.*

Suppose next that B is an associative H-space on which A acts on both sides; i.e., the left and right actions of A commute with each other and are associative, and the 0-cell of A acts like the identity. (If B is an H-complex, then the product of a cell in B by a cell in A is not required to be a cell of B.) Then by a map

$$f : \mathscr{M}_k(X) \to B$$

of $\mathscr{M}_k$ into B, we mean a tuple of two-sided invariant maps $(f_0, \cdots, f_k)$, where f_p takes $M_p(X)$ to B, and

$$f_p = \beta(f_q \times f_r), \qquad p = q + r,$$

β being the multiplication in B.

THEOREM (3.9). *Suppose that*

$$\mathscr{M}_k(X) = (M_0(X), \cdots, M_k(X), \mu_{p,q}, A)$$

is a set of two-sided A-relations on X, and suppose that

$$f : \mathscr{M}_k(X) \to B$$

is a map of the set $\mathscr{M}_k(X)$ into the associative H-space B on which A acts on both sides. Then there is a two-sided invariant homomorphism,

$$f_\infty : M(X) \to B,$$

uniquely determined by f, where $M(X)$ is the RPT-complex generated by X modulo $\mathcal{M}_k(X)$. Moreover, f_∞ is cellular if f is cellular and B is an H-complex.

The proof is straightforward.

In the first part of the section we indicated how the absolute and relative relations correspond to each other. Suppose

$$\mathcal{M}_2(X) = (M_0(X), M_1)X, (M_2(X); \mu_{p,q})$$

is a set of absolute relations; i.e., $M_0(X) = \{a^0\}$ and $M_1(X) = X$. Assume also that $M_2(X) = X_2 \cup D^n$, where X_2 is the 2-fold reduced product of X and D^n is the n-disc. Then, as above, $\mathcal{M}_2(X)$ corresponds to a set of A-relations

$$\mathcal{M}_2(\tilde{X}) = (M_0(\tilde{X}), M_1(\tilde{X}), M_2(\tilde{X}); \tilde{\mu}_{p,q}, A)$$

where $A = X_\infty$, $M_1(\tilde{X}) = A \cup A \times D^n \times A$, and $M_2(\tilde{X}) = [M_1(\tilde{X}), M_1(\tilde{X})]_A$. Here $M_1(\tilde{X})$ is formed by attaching $A \times D^n \times A$ to A by the multiplication $A \times A \times A \to A$. One can see easily that the RPT-complex $M(X)$ generated by X modulo $\mathcal{M}_2(X)$ is naturally isomorphic to the RPT-complex $M(\tilde{X})$ generated by $\tilde{X}$ modulo $\mathcal{M}_2(\tilde{X})$.

4. **The universal quasi-fibration of an RPT-complex.** One knows that an associative H-space A always has a principal quasi-fibration

$$p_A: E(A) \to B(A)$$

such that the total space $E(A)$ is aspherical [1]. (Recall that such a principal quasi-fibration is called *universal*.) This quasi-fibration is, however, not quite appropriate to the needs of this study, since it is not possible in general to tell if the restriction of a quasi-fibration to a subset of the base is also a quasi-fibration. The aim of this section is to construct a universal quasi-fibration (for any RPT-complex) which is more appropriate in that respect for the cases under consideration here (see Corollary (4.5) below). The construction is based on the following theorem.

Suppose A is an RPT-complex of any rank (see §1), and assume that

$$\bar{f}: \bar{L} \to A$$

is a cellular map of some special complex $\bar{L}$ into A. By definition, let

$$M_0(X) = A, \text{ and } M_1(X) = X = A \cup A \times A' \times A,$$

where $A' = A \cup_{\bar{f}} T\bar{L}$($T\bar{L}$ being the cone on $\bar{L}$) and $A \times A' \times A$ is attached to A by the multiplication $A \times A \times A \to A$. Clearly $M_0(X)$ and $M_1(X)$ are two-sided A-complexes; and, if we put $M_2(X) = [M_1(X), M_1(X)]_A$, then the set

$$\mathcal{M}_2(X) = (M_0(X), M_1(X), M_2(X); \mu_{p,q}, A), \tag{4.1}$$

where $\mu_{p,q}$ are the natural maps, is a set of A-relations in the sense of §3. Suppose $M(X)$ is the RPT-complex generated by X modulo $\mathscr{M}_2(X)$. Assume now that $E(A)$ is a contractible right A-complex, and denote $E(A)//A$ by $B(A)$. Then

$$p_A : E(A) \to B(A)$$

(p_A being the natural projection) is a principal quasi-fibration according to Theorem (2.2). We wish to imbed $E(A)$ in a contractible left $M(X)$-complex $E(M)$ in such a way that the action of M on $E(M)$ extends the action of A on $E(A)$. Consider, therefore, the right M-complex $E(A) \times_A M$, and let

$$p'_A : E(A) \times_A M \to B(A)$$

be the quasi-fibration induced by the projection on the first factor. Observe that we can extend $\tilde{f}$ to a map

$$g : L = S \# \tilde{L} \to E(A) \times_A M$$

which is equal to the natural map g_+ of $T\tilde{L}$ into $M \subset E(A) \times_A M$ on the upper cone and to a contraction of $\tilde{f}$,

$$g_- : (T\tilde{L}, \tilde{L}) \to (E(A), A),$$

on the lower cone. ($S \# L$ is the suspension of L.) Then

$$E(M) = E(A) \times_A M \cup_\beta TL \times M, \tag{4.2}$$

where the attaching map,

$$\beta : L \times M \to E(A) \times_A M,$$

is the map that takes (x, y) to $g(x)y$ and is clearly a right M-complex, and the natural projection,

$$p_M : E(M) \to B(M) = E(M)//M$$

is a principal quasi-fibration for M. Observe also that

$$B(M) = B(A) \cup_\gamma TL, \tag{4.3}$$

where γ is the composition of g and p'_A.

THEOREM (4.4). *Suppose $E(A)$ is a contractible right A-complex where A is a given RPT-complex, and let M be the RPT-complex relative to A which is determined by* (4.1). *Then the complex $E(M)$ of* (4.2) *is a contractible right M-complex which contains $E(A)$ as a subcomplex, the imbedding being of a right A-complex into a right M-complex. Moreover, in the induced diagram*

$$\begin{array}{ccc} E(A) & \longrightarrow & E(M) \\ p_A \downarrow & & \downarrow p_M \\ B(A) & \longrightarrow & B(M) \end{array},$$

the vertical maps are universal quasi-fibrations, and $B(M)$ is obtained from $B(A)$ according to (4.3).

We need prove only that $E(M)$ is contractible, which we shall do a bit later.

COROLLARY (4.5). *Suppose A is an RPT-complex. Then there is a universal quasi-fibration,*

$$p_A : E(A) \to B(A).$$

Moreover, if N is an RPT-subcomplex of A, then $E(A)$, $B(A)$ and p_A can be constructed so that

$$p_A^{-1}(K) = E(N) \times_N A,$$

where K is a subcomplex of $B(A)$ and

$$(p_A | E(N)) : E(N) \to K$$

is a universal quasi-fibration for N.

The corollary can be proved easily by expressing A as the ascending sequence of RPT-complexes

$$^2A \subset {}^3A \subset \cdots \subset {}^kA \subset \cdots,$$

where $^2A = X_\infty$, for some special complex (X_∞ being the reduced product of James), and ^kA is obtained from ^{k-1}A by adding one indecomposable cell to ^{k-1}A, in the same manner that the *RPT*-complex $M(X)$ is obtained from A by adding the cone on $\bar{L}$ (see (4.1)). Now, by definition, let

$$E(^2A) = {}^2A \cup TX \times {}^2A,$$

where TX is the cone on X and $TX \times {}^2A$ is attached by $^2\alpha | X \times {}^2A$ ($^2\alpha$ being the multiplication in ^2A). It is easy to see that $E(^2A)$ is contractible; and, if we make ^2A act on the right on $E(^2A)$, then $E(^2A)$ becomes a right ^2A-complex, and

$$p_2 : E(^2A) \to E(^2A) // {}^2A = B(^2A) = S \# X$$

is a universal quasi-fibration for ^2A. Now, by Theorem (4.4), we can find an ascending sequence

$$\begin{array}{ccc} E({}^2A) \subset \cdots \subset & E({}^kA) & \subset \cdots \\ \downarrow p_2 & \downarrow p_k & \\ B({}^2A) \subset \cdots \subset & B({}^kA) & \subset \cdots, \end{array}$$

where p_k is a universal quasi-fibration for ${}^kA$.

By definition, let

$$E(A) = \lim_k \; E({}^kA), \; B(A) = \lim_k \; B({}^kA), \text{ and } p_A = \lim_k p_k.$$

Then the map

$$p_A: E(A) \to B(A)$$

is a universal quasi-fibration for A (see (2.1) of [2]).

The fact that $E(A)$ can be constructed so as to satisfy the second assertion of the corollary can be shown by constructing $E(N)$ first and then applying Theorem (4.4) to a filtration of A by RPT-subcomplexes starting with N and such that each RPT-subcomplex is obtained from the preceding one by adding one indecomposable cell only.

Similarly one can construct a universal quasi-fibration on which A acts on the left.

Proof of Theorem (4.4). As we mentioned above, we need prove only that $E(M)$ is contractible. So let $P(B(M))$ be the space of Moore paths based at the 0-cell b^0, and let

$$q: P(B(M)) \to B(M)$$

be the fibration that sends a path to its end-point. The fiber is the space of loops $\Omega(B(M))$. Observe that $\Omega(B(M))$ is an associative H-space and that it acts on the right on $P(B(M))$. We can find a homomorphism,

$$u: M \to \Omega(B(M)),$$

which is a homotopy equivalence [3]. It is not hard to see that u can be extended to a map

$$v: E(M) \to P(B(M))$$

of an M-space into an $\Omega(B(M))$-space such that the induced diagram

$$\begin{array}{ccc} E(M) & \xrightarrow{\quad v \quad} & P(B(M)) \\ \downarrow p_M & & \downarrow q \\ B(M) & \xrightarrow{\quad \bar{v} \quad} & B(M) \end{array}$$

is commutative and the induced map $\bar{v}$ is homotopic to the identity. This implies that $E(M)$ is contractible, which is what is to be proved.

REMARK (4.6). Given a simply-connected complex B, one can find an RPT-complex A and a homomorphism ϕ of A into $\Omega(B)$ which is a homotopy equivalence. It is easy to see that

$$B \sim B(A).$$

Thus the universal quasi-fibration represents the canonical fibration of the space of paths over B.

Next we show that the principal quasi-fibrations

$$p: N \to N//A$$

admits a *classifying* map, where N is a right A-complex. Let

$$p_A: E(A) \to B(A)$$

be a universal quasi-fibration on which A acts on the *left*, constructed as above. Consider the commutative diagram,

$$\begin{array}{ccccc} N & \xleftarrow{\beta'} & N \times E(A) & \xrightarrow{\pi'} & E(A) \\ \downarrow{\scriptstyle p} & & \downarrow & & \downarrow{\scriptstyle p_A} \\ N//A & \xleftarrow{\beta} & N \times_A E(A) & \xrightarrow{\pi} & B(A) \end{array}$$

where the vertical maps are the natural projections and the horizontal maps are the projections on the factors and the maps induced by them.

Theorem (2.2)′ implies that

$$\beta: N \times_A E(A) \to N//A \quad \text{and}$$

$$\pi: N \times_A E(A) \to B(A)$$

are quasi-fibrations with fibers $E(A)$ and N respectively. Since $E(A)$ is contractible, β has a homotopy-inverse, β^{-1}, say. Define

$$\chi = \beta^{-1}\pi: N//A \to B(A) \tag{4.7}$$

to be the *classifying map* of the quasi-fibration

$$p: N \to N//A.$$

5. **Fibrations of complexes by subcomplexes.** Let us say that a sequence

$$\mathscr{F}: F \xrightarrow{i} B \xrightarrow{f} Y$$

of special complexes and cellular maps is a *fiber sequence*, if and only if, $\mathscr{F}$ can be imbedded in a homotopy-commutative diagram,

$$\begin{array}{ccccc} F & \xrightarrow{i} & B & \xrightarrow{f} & Y \\ \downarrow & & \downarrow & & \downarrow \\ F' & \xrightarrow{i'} & B' & \xrightarrow{f'} & Y', \end{array}$$

where the lower row is a Serre fibration and the vertical maps are all homotopy-equivalences. F is the *fiber*, B the *total space*, and Y the *base-space* of $\mathscr{F}$.

Consider now the following problem, which is essentially Problem 10 in Massey's list of 1955 [6]: *suppose*

$$i: F \to B$$

is a map of special complexes. When is it possible to imbed i in a fiber sequence

$$\mathscr{F}: F \xrightarrow{i} B \xrightarrow{f} Y?$$

Is $\mathscr{F}$ unique?

Assume i is an imbedding, since there is no loss of generality in doing so. One can see right away that the problem does not always have a solution. For example, let $F = S^{2n}$, the $2n$-sphere with $n > 0$, $B = S^{2n+1}$, the $(2n+1)$-sphere, and let $i: S^{2n} \to S^{2n+1}$ be the natural imbedding. Since S^{2n} cannot be an H-space, i cannot be imbedded in a fiber sequence [10]. Note also that the injection of F in B plays an important role. In fact, for some spaces F and B it may happen that one imbedding gives rise to a fiber sequence, while another does not. For example, let $B = TF \times F$, where TF is the cone on F. Then if we imbed F in B as the second factor, the projection on the first factor defines a fiber sequence for which F is the fiber; but if we imbed F as the base of the second factor TF, then the resulting sequence is not a fiber sequence, for this would imply that F was an H-space [10].

Moreover, the sequence

$$\mathscr{F}: F \xrightarrow{i} B \xrightarrow{f} Y$$

in which i can be imbedded is not unique. For example, let $F = S^3$, the 3-sphere, and $B = S^7$, the 7-sphere; and let i be the natural imbedding. Then there are inequivalent multiplications $\{\alpha\}$, on S^3, which yield inequivalent fiber sequences

$$\alpha: S^3 \to S^7 \to S^4,$$

according to [5].

We shall give necessary and sufficient conditions for the problem to have a solution, as well as a method, at least in theory, for obtaining all solutions when B and F are *simply connected special complexes* and F is a *subcomplex* of B. Denote the space of Moore paths based at b^0 by $P(B)$ and by $\Omega(B)$ the subspace of loops. Then, with the usual juxtaposition multiplication, $P(B)$ is a right $\Omega(B)$-space. Suppose

$$u: A \to \Omega(B) \tag{5.1}$$

is a homomorphism of some RPT-complex A into $\Omega(B)$, which is a homotopy equivalence [3]. Then, as we saw in §4, there is a commutative diagram,

$$\begin{array}{ccccc} A & \longrightarrow & E(A) & \xrightarrow{p_A} & B(A) \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle \bar{v}} \\ \Omega(B) & \longrightarrow & P(B) & \xrightarrow{q} & B \end{array},$$

where the upper row is the universal quasi-fibration for A and the lower row is the canonical fibration which sends a path to its end-point and v is a map of a right A-space into a right $\Omega(B)$-space. Moreover, u, v, and $\bar{v}$ are all homotopy equivalences. Now, let

$$U = p_A^{-1}\bar{v}^{-1}(F), \text{ and } \Omega(B,F) = q^{-1}(F).$$

Corollary (4.5) of §4 implies that the restriction of p_A to U is a quasi-fibration, and hence, comparing the homotopy groups, we conclude that

$$(v \mid p_A^{-1}\bar{v}^{-1}(F)): U \to \Omega(B,F) \tag{5.2}$$

is a homotopy equivalence. (Recall that $\Omega(B,F)$ is of the homotopy type of a countable CW-complex [8].)

What kind of A-relations does U admit, when F is the fiber for some fiber sequence in which the total space is B? The following theorem gives the answer.

THEOREM (5.3). *Suppose that*

$$F \xrightarrow{i} B \xrightarrow{f} Y$$

is a fiber sequence such that B and F are simply connected special complexes and F is a subcomplex imbedded by i in B. Assume also that A is the RPT-complex representing $\Omega(B)$ and U the right A-complex representing $\Omega(B,F)$ as in (5.1) and (5.2). Then there is a right invariant map,

$$\phi: U \to M_1,$$

which is a homotopy equivalence, and, for every $n \geq 1$, there is a set of A-relations on M_1,

$$\mathcal{M}_n(M_1) = (M_0, \cdots, M_n; \mu_{p,q}, A),$$

such that

$(a)_n$ *$\mathcal{M}_k(M_1)$ is a subset of $\mathcal{M}_n(M_1)$, for $k < n$, and*

$(b)_n$ *there is a two-sided invariant deformation retraction,*

$$^k h_t : M_k \to M_k, \ k = 2, \cdots, n,$$

of M_k onto M_{k-1}.

The proof will be given in §7 below.

DEFINITION (5.4). *Suppose that M_1 is a two-sided A-complex, A being an RPT-complex. Then M_1 is said to admit a Q_n-structure if and only if, there is a set of A-relations on M_1,*

$$\mathcal{M}_n(M_1) = (M_0, \cdots, M_n; \mu_{p,q}, A),$$

subject to condition $(b)_n$ of Theorem (5.3).

Note that the necessary conditions of Theorem (5.3), which are satisfied if F is the fiber of a fiber sequence for which B is the total space, are of two kinds. First, U or, equivalently, $\Omega(B,F)$ must be of the homotopy type of a two-sided A-complex M_1; and second, M_1 must admit a Q_n-structure for all $n \geqq 2$, and the Q_n-structures must extend each other. We shall take up the discussion of when M_1 exists in another paper, confining our attention here to the implications of a Q_n-structure. We shall see in effect that the existence of a Q_n-structure is a sufficient condition for the existence of sequences,

$$\mathcal{F}_n: F \xrightarrow{i} B \xrightarrow{f} B_n,$$

which are fiber sequences in a range which widens with n. The following condition will be needed in most of what follows.

CONDITION (5.5). *Suppose B and F are simply connected special complexes with F a subcomplex of B, and let A and U be the RPT-complex and the right A-complex representing $\Omega(B)$ and $\Omega(B,F)$, as in (5.1) and (5.2). The condition is that there be a two-sided A-complex M_1 and a right A-map*

$$\phi: U \to M_1$$

which is a homotopy equivalence.

Consider now the set of A-relations

$$\mathcal{M}_n(M_1) = (M_0, \cdots, M_n; \mu_{p,q}, A), \tag{5.6}$$

and denote by ^kM the RPT-complex relative to A which is generated by M_1 modulo the subset $\mathcal{M}_k(M_1)$ of $\mathcal{M}_n(M_1)$. Put $B_k = B(^kM)$, the base-space of the universal quasi-fibration of ^kM (see §4). Then the imbedding of ^{k-1}M into ^kM induces an imbedding

$$\eta_k : B_{k-1} \to B_k.$$

Thus we obtain an ascending sequence

$$B \to B_1 \to \cdots \to B_n$$

of special complexes. Let

$$f_k = \eta_k \cdots \eta_1 : B \to B_k$$

and

$$F_k = \Omega(B_k, B) = \{\alpha \in \tilde{P}(B_k) \mid \alpha(0) \in B\},$$

where $\tilde{P}(B_k)$ is the space of Moore paths α ending in b^0 (i.e., $\alpha(1) = b^0$). Suppose that

$$i_k : F_k \to B$$

is the natural map which sends a path to its initial point. Then

$$\begin{array}{ccccc}
F_1 & \xrightarrow{i_1} & B & \xrightarrow{f_1} & B_1 \\
\lambda_2 \downarrow & & \downarrow \| & & \downarrow \eta_2 \\
F_2 & \xrightarrow{i_2} & B & \xrightarrow{f_2} & B_2 \\
\lambda_3 \downarrow & & \downarrow \| & & \downarrow \eta_3 \\
\cdots & \longrightarrow & \cdots & \longrightarrow & \cdots \\
\lambda_n \downarrow & & \downarrow \| & & \downarrow \eta_n \\
F_n & \xrightarrow{i} & B & \xrightarrow{f_n} & B_n
\end{array}$$

is a commutative diagram in which the rows are fiber sequences, and λ_j is the natural imbedding of F_{j-1} into F_j.

THEOREM (5.7). *Suppose that Condition* (5.5) *is satisfied. Assume that the set of A-relations* (5.6) *defines a Q_n-structure on M_1, the two-sided A-complex of* (5.5). *Then, for $k \leqq n-1$, with the notation as in the preceding paragraph, there are natural maps,*

$$\gamma_k : F \to F_k \text{ and } \beta_k : F_k \to F,$$

for $k = 1, \cdots, n$, such that

(a)$_k$ $\beta_k \gamma_k \sim 1$, $\lambda_k \gamma_k \sim \gamma_{k+1}$, *and $gi = i_k \gamma_k$, where g is a homotopy equivalence of B; and*

(b)$_k$ *the induced homomorphism*

$$(\gamma_k)_* : \pi_j(F) \to \pi_j(F_k)$$

is an isomorphism for $j \leqq c(k+1)+(k-2)$, *where* $(c-1)$ *is the connectivity of* F. ($\sim$ *denotes homotopy.*)

The proof is given in §9.

Thus the existence of a Q_n-structure on M_1 implies that the pair (B,F) can be "approximated" by pairs (B,F_n) with the property that F_n is a fiber in a fiber sequence in which B is the total space and F and F_n agree in a range which increases with n. Thus the existence of Q_n-structures on $\Omega(B,F)$ which extend each other for all n is a sufficient condition for the existence of a fiber sequence

$$F \to B \to X.$$

Note that if B is contractible, then the fibers F_k are all of the homotopy type of loopspaces [9], and, hence, F is increasingly getting like a loopspace. This suggests that if B is contractible, our conditions are just Stasheff's A_n-structures [11]. In fact, this is the case. First we need prove that Condition (5.5) is satisfied.

(5.8). *Suppose B is a contractible special complex and F is a simply connected subcomplex, and let A and U be the RPT-complex and the right A-complex representing* $\Omega(B)$ *and* $\Omega(B,F)$, *as in* (5.1) *and* (5.2).

Then there is a right A-map,

$$\phi: U \to M_1,$$

of U into a two-sided A-complex M_1 which is a homotopy equivalence.

In fact, let

$$M_1 = U \cup_\beta A \times U,$$

where the attaching map is the map

$$\beta: A \times A \cup U \to U$$

induced by the multiplication in A. Then clearly M_1 is a two-sided A-complex, and the right A-imbedding,

$$\phi: U \to M_1,$$

is a homotopy equivalence.

Let us denote by $\bar{M}_1$ the complex M_1 without the action of A. (Recall that an A_n-structure in the sense of Stasheff is according to [3], just a Q_n-structure relative to $\{a^0\}$, a^0 being the 0-cell of A.)

THEOREM (5.9). *Suppose that the assumptions of* (5.8) *are satisfied, and let $\bar{M}_1$ and M_1 be as above. Assume that*

$$\mathscr{M}_n(\bar{M}_1) = (\bar{M}_0, \cdots, \bar{M}_n; \bar{\mu}_{p,q}, \{a^0\})$$

is a set of $\{a^0\}$*-relations which defines an* A_n*-structure on* $\bar{M}_1$*. Then there is a set of* A*-relations*

$$\mathscr{M}_n(M_1) = (M_0, \cdots, M_n; \mu_{p,q}, A)$$

which defines a Q_n*-structure on* M_1*, and conversely. In either case, there is a map*

$$f_n : \mathscr{M}_n(\bar{M}_1) \to \mathscr{M}_n(M_1)$$

which is a homotopy equivalence when restricted to $\bar{M}_1$*.*

We shall sketch the proof in the case $n = 2$ only, the general case being similar. So suppose that

$$\mathscr{M}_2(M_1) = (M_0, M_1, M_2; \mu_{p,q}, A)$$

is a set of A-relations which defines a Q_2-structure on M_1 (see Definition (5.4)). By definition, let

$$\bar{M}_0 = \{a^0\} \text{ and } \bar{M}_1 = M_1,$$

and let

$$\gamma : [\bar{M}_1, \bar{M}_1] \to [M_1, M_1]_A$$

be the natural projection. Let

$$\bar{M}_2 = M_2 \cup_\gamma [\bar{M}_1, \bar{M}_1] \times I,$$

where I is the unit interval, the point $(x, 1)$ is identified with $\gamma(x) \in M_2$, and $\{a^0\} \times I$ is identified with the basepoint a^0. ($\bar{M}_2$ is just the mapping cylinder of γ with the segment through a^0 identified with a^0.) Clearly the set

$$\mathscr{M}_2(\bar{M}_1) = (\bar{M}_0, \bar{M}_1, \bar{M}_2; \bar{\mu}_{p,q}, \{a^0\})$$

where $\bar{\mu}_{p,q}$ are the natural maps is a set of relations which defines an A_2-structure on $\bar{M}_1$, and the natural map

$$f_2 : \mathscr{M}_1(\bar{M}_2(\bar{M}_1) \to \mathscr{M}_2(M_1)$$

is a map which satisfies the requirements of the theorem.

The converse is similar. Suppose that

$$\mathscr{M}_2(\bar{M}_1) = (\bar{M}_0, \bar{M}_1, \bar{M}_2; \bar{\mu}_{p,q}, \{a^0\})$$

is a set of $\{a^0\}$-relations which defines an A_2-structure on $\bar{M}_1$. By definition

$$M_0 = A \text{ and } M_1 = \bar{M}_1.$$

Suppose also that γ is the natural map

$$\gamma : [\bar{M}_1, \bar{M}_1] \to [M_1, M_1]_A.$$

Then since A is contractible (B being so), we can easily check that the natural map

$$\bar{M}_1 \times \bar{M}_1 \to M_1 \times_A M_1$$

is a homotopy equivalence. This implies that γ is a homotopy equivalence by comparing the homology groups. Hence

$$\tilde{M}_2 = [M_1, M_1]_A \cup_\gamma \bar{M}_2,$$

where $\bar{M}_2$ is attached to $[M_1, M_1]_A$ by γ, is of the homotopy type of M_2, and the two-sided A-complex

$$M_2 = \tilde{M}_2 \cup A \times \tilde{M}_2 \times A,$$

where $A \times \tilde{M}_2 \times A$ is attached to $\tilde{M}_2$ by the map

$$A \times [M_1, M_1]_A \times A \cup \tilde{M}_2 \to \tilde{M}_2,$$

is of the homotopy type of M_1. To find an invariant two-sided deformation retraction of M_2 onto M_1, apply Lemma (7.1) of §7. Thus, with $\mu_{p,q}$ being the natural maps,

$$\mathscr{M}_2(M_1) = (M_0, M_1, M_2; \mu_{p,q}, A)$$

is a set of A-relations which defines a Q_2-structure on M_1.

Observe that the first part of the proof did not use the fact that B is contractible. In fact, the same argument can show that any set of A-relations on a two-sided A-complex M_1 defines a set of A'-relations on M_1. A' being any RPT-subcomplex of A_1. The two sets are related but do not define the same RPT-complex.

6. **Fibrations of the Hopf type defined by a Q_n-structure.** One of the attractive features of the Hopf fibrations defined by an H-structure on a space is that they enable one to reduce the property of the existence of an H-structure, which is essentially an algebraic property, to the existence of fibrations and projective spaces defined by them (see e.g., [9]). In fact, this has been one of the most productive tools in the study of H-spaces. We shall see that the A-relations defining a Q_n-structure provide us with fibrations on the Hopf type. First we need introduce the concept of smashed product and join in the category of A-spaces. Suppose M_1 is a two-sided A-complex, and let ${}^2M$ be the RPT-complex generated by M_1 modulo the set of A-relations

$$\mathscr{M}_2(M_1) = (M_0, M_1, [M_1, M_1]_A; \mu_{p,q}, A)$$

where $\mu_{p,q}$ are the natural maps. Consider

$$[\ ]_A^n(M_1) = [M_1, \cdots, M_1]_A,$$

the subcomplex of ${}^2M$ of index n. Then $[\ ]_A^n(M_1)$ is a two-sided A-complex which includes $[\ ]_A^{n-1}(M_1)$ as a two-sided A-subcomplex. By definition, the *n-fold smashed product of $M_1//A$ relative to A is*

$$(6.1)\qquad (\#)_A^n(M_1//A) = ([\ ]_A^n(M_1)//A)/([\ ]_A^{n-1}(M_1)//A),$$

the special complex obtained from $[\ ]_A^n(M_1//A)$ by identifying the subcomplex $[\ ]_A^{n-1}(M_1//A)$ with the basepoint. Note that if $A = \{a^0\}$, where a^0 is a single 0-cell, then

$$(\#)_A^n(M_1//A) = (\#)^n(M_1),$$

the usual smashed product, since in this case $M_1//A = M_1$. By definition, the *n-fold join of $M_1//A$ relative to A* is

$$(6.2)\qquad (*)_A^n(M_1//A) = S^{n-1} \# (\#)_A^n(M_1//A),$$

where $S^{n-1} \# (\#)_A^n(M_1//A)$ is the *usual* smashed product of the $(n-1)$-sphere and $(\#)_A^n(M_1//A)$. Again observe that if $A = \{a^0\}$, then the join relative to $\{a^0\}$ is just the usual join.

Suppose that M_1 is a two-sided A-complex, and let

$$\mathcal{M}_n(M_1) = (M_0, \cdots, M_n; \mu_{p,q}, A)$$

be a set of relations which defines a Q_n-structure on M_1 (see Definition (5.4) above). Denote by ${}^kM$ the RPT-complex generated by M_1 modulo the subset $\mathcal{M}_k(M_1)$, $k = 2, \cdots, n$, and by B_k, the basespace of the universal quasi-fibration of ${}^kM$ (see §4). We have then the following theorem.

THEOREM (6.3). *There is a commutative diagram,*

$$\begin{array}{ccccc}
M_1 & \longrightarrow & (*)_A^1(M_1//A) & \longrightarrow & B \\
\downarrow \| & & \downarrow & & \downarrow \eta_1 \\
\cdots & & \cdots & & \cdots \\
\downarrow \| & & \downarrow & & \downarrow \\
M_1 & \longrightarrow & (*)_A^k(M_1//A) & \longrightarrow & B_{k-1} \\
\downarrow \| & & \downarrow & & \downarrow \eta_k \\
\cdots & \longrightarrow & \cdots & \longrightarrow & \cdots \\
\downarrow \| & & \downarrow & & \downarrow \eta_{n-1} \\
M_1 & \longrightarrow & (*)_A^n(M_1//A) & \longrightarrow & B_{n-1},
\end{array}$$

where the rows are fiber sequences, the maps η_k are those induced by the imbed-

dings ${}^{k-1}M \to {}^{k}M$, *and* $(*)^k_A(M_1//A)$ *is the k-fold join of* $M_1//A$ *relative to* A. *Moreover,* M_1 *is contractible in* $(*)^k_A(M^k_1//A)$, *for* $k \geqq 2$, *and* $(*)^k_A(M_1//A)$ *is contractible in* $(*)^{k+1}_A(M_1//A)$, *for* $k \geqq 1$.

The proof will be given in §8 below. Note that if $A = \{a^0\}$, then the preceding fibrations are the usual Hopf fibrations associated with an A_n-structure [11].

Suppose now that Condition (5.5) is satisfied, and assume

$$\mathcal{M}_n(M_1) = (M_0, \cdots, M_n; \mu_{p,q}, A)$$

is a set of A-relations which defines a Q_n-structure on M_1. Then Theorem (6.3) implies that there is a ladder of fiber sequences

$$\begin{array}{ccccccc} M_1//A & \subset & M_1//A *_A M_1//A & \subset \cdots \subset & (*)^n_A(M_1//A) \\ \downarrow & & \downarrow & \downarrow & \downarrow \\ B & \subset & B_1 & \subset \cdots \subset & B_{n-1} \end{array}$$

with fiber M_1. Observe that, from the second stage on, M_1 is contractible in the total space $(*)^k_A(M_1//A)$, while $(*)^k_A(M_1//A)$ is always contractible in $(*)^{k+1}_A(M_1//A)$. This implies that

$$B_k \sim B_{k-1} \cup T((*)^k_A(M_1//A)) \cup \text{other cells}, \tag{6.4}$$

where $T((*)^k_A(M_1//A))$ is the cone on $(*)^k_A(M_1//A)$. The fact that ϕ (see (5.5)) is a right invariant map which is a homotopy equivalence implies that the induced map

$$\bar{\phi}: U//A = F \to M_1//A$$

is also a homotopy equivalence. But what can we say about $(*)^k_A(M_1//A)$, for $k \geqq 2$? A partial answer is given by the following proposition.

PROPOSITION (6.5). *Suppose F is a suspension. Then there is a natural homotopy equivalence*

$$w: (*)^k_A(M_1//A) \to (*)^k(F), \ k \geqq 1.$$

PROOF. Suppose that

$$F = S \# \bar{F},$$

where $\bar{F}$ is a special complex. Since F is a subcomplex of B, $\bar{F}_\infty$ is an RPT-subcomplex of A, the RPT-complex representing $\Omega(B)$ as in (5.1) above. This means that

$$U = E(\bar{F}_\infty) \times_{\bar{F}_\infty} A,$$

where $E(\bar{F}_\infty)$ is the universal quasi-fibration of $\bar{F}_\infty$ (see Corollary (4.5)), and

$$E(\bar{F}_\infty) = \bar{F}_\infty \cup T\bar{F} \times \bar{F}_\infty$$

(see proof of Corollary (4.5)). Hence

$$\bar{U} = A \cup T\bar{F}$$

where the cone $T\bar{F}$ is attached by the imbedding on the base is a subcomplex of U. Note that

$$\bar{U}/A = U//A,$$

and hence $\bar{U}/A$ and $M_1//A$ are of the same homotopy type. To finish the proof, note that the imbedding of $\bar{U}$ induces a map,

$$t:(\bar{U}/A)\#(\bar{U}/A) \to (M_1//A)\#_A(M_1//A).$$

Since U is right invariant homotopy equivalent to M_1 (see Remark (7.2) of §7), it follows that $(U//A)\#_A(M_1//A)$ (defined by identifying the image of $U \times A \cup A \times M_1$ in $(U \times_A M_1)//A$ with the basepoint) is homotopy-equivalent to $(M_1//A)\#_A(M_1//A)$. But

$$(U//A)\#_A(M_1//A) = (\bar{U}/A)\#(M_1//A),$$

since $U = \bar{U} \cup \bar{U} \times A$, the attaching map being the multiplication

$$A \times A \to A \subset \bar{U}.$$

This is enough to prove that t is a homotopy equivalence and, therefore, the proposition when $k = 2$. The general case follows similarly by induction.

Rewriting (6.4), we get the following

$$(6.4)' \qquad B({}^kM) \sim B({}^{k-1}M) \cup T((*)^k_A F) \cup \text{other cells},$$

when F is a suspension. How are the "other cells" related to F and B? This turns out to be a tractable problem. A complete description of how they are formed out of F and B and how they are attached to $B({}^{k-1}M) \cup T((*)_A F)$, as well as some applications of these results, will be given in a future article. Here we shall note only that, if B is contractible, then these extra cells do not change the homotopy type of $B({}^{k-1}M) \cup T((*)^k_A F)$. To see this, apply Theorem (5.9) of §5 to replace the given set of A-relations,

$$\mathscr{M}_n(M_1) = (M_0, \cdots, M_n; \mu_{p,q}, A),$$

defining the Q_n-structure on $M_1 = \Omega(B, F)$ by a set

$$\mathscr{M}_n(\bar{M}_1) = (\bar{M}_0, \cdots, \bar{M}_n; \mu_{p,q}, \{a^0\})$$

of relations which defines an A_n-structure on $\bar{M}_1 = M_1$. We also have a homomorphism

$$f_k: {}^k\bar{M} \to {}^kM, \qquad k = 1, \cdots, n,$$

of the RPT-complex ${}^k\bar{M}$ generated by $\bar{M}_1$ modulo $\mathscr{M}_k(\bar{M}_1)$ into the RPT-complex ${}^kM$ generated by M_1 modulo $\bar{M}_k(M_1)$. Now we note that f_k induces a map

$$\tilde{f}_k : B({}^k\bar{M}) \to B({}^kM)$$

and that Theorem (6.3) implies that $\tilde{f}_k$ is a homotopy equivalence, which is enough to prove the assertion.

Thus, if B is contractible, $B({}^nM)$ is the projective space of F corresponding to the A_n-structure which is defined by the given Q_n-structure, and conversely.

It is well known that F is an H-space if, and only if, $F_\infty = \Omega S \# F$ retracts onto F (e.g., see [4]). This result was extended to A_n-structures in [3]: F admits an A_n-structure if and only if, $\Omega(P_n(F))$ can be retracted onto F, $P_n(F)$ being the n-projective space of F. The following theorem gives a similar result for Q_n-structures.

THEOREM (6.5). *Suppose M_1 is a two-sided A-complex, A being an RPT-complex, and assume that M_1 admits a Q_n-structure defined by the set of A-relations*

$$\mathscr{M}_n(M_1) = (M_0, \cdots, M_n; \mu_{p,q}, A).$$

Then this Q_n-structure can be extended to a Q_{n+1}-structure if there is a two-sided invariant retraction,

$$r_{n+1} : {}^nM \to M_n,$$

where ${}^nM$ is the RPT-complex generated by M_1 modulo $\mathscr{M}_n(M_1)$ and M_n is the subcomplex of ${}^nM$ of index n.

Conversely, if the given Q_n-structure can be extended to a Q_{n+1}-structure, then there is an invariant right retraction of ${}^nM$ onto M_n.

The fact that a Q_n-structure can be extended to a Q_{n+1}-structure if r_{n+1} exists can be proved in the same way as Theorem (4.6) of [3], the only modification being to replace "complex" by "A-complex" and "map" by "invariant map." To prove the converse, apply Theorem (6.3) to prove that

$$K_\infty \times M_n \text{ and } {}^nM \text{ are of the same homotopy type,}$$

(K being a special complex such that $S \# K = (\#)_A^{n+1}(M_1//A)$), and then apply Remark (7.2) of §7 to the mapping cylinder of the homotopy equivalence to get the right invariant retraction. (Consider $K_\infty \times M_n$ as a right A-complex by making A act on M_n.)

7. **Proof of Theorem (5.3).** In general outline the proof resembles that of Theorem (4.5) of [3], but some of the details are different. Suppose

$$F \xrightarrow{i'} B \xrightarrow{f} Y$$

is a fiber sequence of special complexes, where B and F are simply connected and F is a subcomplex of B. Assume also that B is a suncomplex of Y. As in §5, let

$$g:A \to \Omega(B)$$

be a homomorphism of the RPT-complex A into $\Omega(B)$, which is also a homotopy equivalence, and

$$g':U \to \Omega(B,F),$$

the map of an A-space into an $\Omega(B)$-space, which is a homotopy equivalence (see (5.1) and (5.2)). Now, according to [3], there is an RPT-complex M, possibly of infinite rank, which contains A as a sub-H-complex, and a homomorphism,

$$\eta:M \to \Omega(Y),$$

which extends g and is a homotopy equivalence. Clearly M is a two-sided A-complex.

By definition, let

$$M_0 = A \text{ and } M_1 = M.$$

Next, let

$$\tilde{M}_2 = [M_1, M_1]_A.$$

Then the mapping

$$\tilde{\eta}_2:\tilde{M}_2 \to M_1$$

which sends the point $[x_1,x_2]$ to $\gamma(x_1,x_2)$, where $x_1,x_2 \in \tilde{M}_2$ and γ, the multiplication in M_1, is a two sided invariant retraction of $\tilde{M}_2$ onto M_1. By definition, let

$$M_2 = \tilde{M}_2 \cup_{\beta_2} \tilde{M}_2 \times I$$

where I is the unit interval $[0,1]$ and the attaching map β_2 is defined by the equations

$$\beta_2(x,0) = x, \ \beta_2(x,1) = \tilde{\eta}_2(x), \text{ for } x \in \tilde{M}_2, \text{ and}$$

$$\beta_2(x,t) = x, \text{ for } (x,t) \in M_1 \times I.$$

Now, if we make A act on M_2 according to the rule

$$a(x,t) = (ax,t) \text{ and}$$

$$(x,t)a = (xa,t),$$

for all $(x,t) \in M_2$ and $a \in A$, then M_2 becomes a two-sided A-complex, and the map

$${}^2h_s: M_2 \to M_2,$$

which sends the point (x,t) to $(x,(1-s)t+s)$, is a two-sided invariant deformation retraction of M_2 onto M_1. Hence the set

$$\mathscr{M}_2(M_1) = (M_0, M_1, M_2; \mu_{p,q}, A),$$

where the complexes M_i are defined above and $\mu_{p,q}$ are the natural maps, is a set of A-relations on M_1, which defines a Q_2-structure on M_1 as desired (see Definition (5.4)).

Proceed by induction. Suppose that we have been able to define a set of A-relations

$$\mathscr{M}_{m-1}(M_1) = (M_0, \cdots, M_{m-1}; \mu_{p,q}, A)$$

and a two-sided invariant mapping,

(i) ${}^jh: M_j \times I \to M_j$, $j = 2, \cdots, m-1$,

which is a deformation retraction of M_j onto M_{j-1}, and a two-sided invariant retraction

(ii) $\eta_j: M_j \to M_1$, $j = 2, \cdots, m-1$

such that

$$\gamma(\eta_p \times \eta_q) = \eta_{p+q}\mu_{p,q}, \qquad p,q \geqq 0 \text{ and } p+q > m-1,$$

γ being the multiplication in $M = M_1$. We wish to do the same thing for the case m. Consider, therefore,

$$\tilde{M}_m = \bigcup [M_p, M_q]_A, \text{ with } p+q = m,$$

the subcomplex of index m of the RPT-complex ${}^{m-1}M$ generated by M_1 modulo $\mathscr{M}_{m-1}(M_1)$. Since the mappings η_j and γ are two-sided invariant and associative, the maps η_j induce together a two-sided invariant retraction,

$$\tilde{\eta}_m: \tilde{M}_m \to M_1,$$

such that

$$\tilde{\eta}_m \big| M_k = \eta_k, \text{ with } 1 \leqq k \leqq m-1.$$

By definition, let

$$\bar{M}_m = \tilde{M}_m \cup \tilde{M}_m + I,$$

where I is the unit interval $[0,1]$ and the attaching map β_m is defined by the equations

$$\beta_m(x,0) = x, \beta_m(x,1) = \eta_m(x), \text{ for } x \in \tilde{M}_m, \text{ and}$$

$$\beta_m(x,t) = x, \text{ for } (x,t) \in M_1 \times I.$$

Now, if we make A act on $\bar{M}_m$ by the rule

$$a(x,t) = (ax,t) \text{ and}$$

$$(x,t)a = (xa,t),$$

for all $(x,t)\in \bar{M}_m$, then $\bar{M}_m$ becomes a two-sided A-complex. Define

$$\eta_m : \bar{M}_m \to M_1$$

to be the map that takes (x,t) to $\tilde{\eta}_m(x)$. Clearly the set

$$\mathscr{M}_m(\bar{M}_1) = (\bar{M}_0, \cdots, \bar{M}_m; \bar{\mu}_{r,q}, A),$$

where $\bar{M}_k = M_k$, for $k < m$, and $\bar{\mu}_{p,q}$ are the natural maps, is a set of A-relations which satisfies (ii) of the induction hypothesis and which extends $\mathscr{M}_{m-1}(M_1)$. Thus, to finish the induction step, we need find a two-sided invariant map

$$^m h : \bar{M}_m \times I \to \bar{M}_m$$

which deforms $\bar{M}_m$ onto $\bar{M}_{m-1}$.

Note that the imbedding

$$\gamma : \bar{M}_{m-1} \to \bar{M}_m$$

is a homotopy equivalence, since each of the complexes is of the homotopy type of $\bar{M}_1 = M_1$. Hence we need prove the following lemma only.

LEMMA (7.1). *Suppose N and N' are two-sided A-complexes, and assume that there is an imbedding*

$$\gamma : N \to N'$$

which is a homotopy equivalence. Then there is a two-sided invariant deformation retraction

$$\phi_t : N' \to N'$$

of N' onto N.

Proof. According to §1, there is an ascending sequence of two-sided A-complexes,

$$N = N_0 \subset \cdots \subset N_k \subset \cdots \subset N',$$

such that $N' = \bigcup_k N_k$ and, for each $k \geqq 1$,

$$N_k = \tilde{N}_k \cup A \times \tilde{N}_k \times A,$$

where $\tilde{N}_k$ is the subcomplex

$$\tilde{N}_k = N_{k-1} \cup_{f_k} D^{n_k}$$

(D^{n_k} being the n_k-disc attached to N_{k-1} by some map f_k) and the complex $A \times \tilde{N}_k \times A$ is attached to $\tilde{N}_k$ by the map

$$A \times N_{k-1} \times A \cup \tilde{N}_k \to \tilde{N}_k,$$

which is induced by the multiplication by A. We shall construct the deformation

ϕ_t by induction on the two-sided A-subcomplexes N_k. Now, since the imbedding γ of N into N' is a homotopy equivalence, the attaching map

$$f_1:(D^{n_1}, S^{n_1-1}) \to (N', N)$$

can be deformed $\operatorname{rel} S^{n_1-1}$, into a map of D^{n_1} into N. This deformation yields a deformation retraction,

$$(\tilde{\phi}_t)_1 : \tilde{N}_1 \to N',$$

of $\tilde{N}_1$ onto N which is extendable uniquely to a two-sided invariant deformation retraction,

$$(\phi_t)_1 : N_1 \to N',$$

of N_1 onto N. Suppose that, by induction, we have been able to find a two-sided A-deformation retraction,

$$(\phi_t)_{m-1} : N_m \to N',$$

of N_m onto N. Consider the characteristic map

$$f : (D^{n_m}, S^{n_m-1}) \to (\tilde{N}_m, N_{m-1}).$$

Note that we have a deformation of S^{n_m-1} into N defined by $(\phi_t)_{m-1}$. By the homotopy extension property of the pair (D^{n_m}, S^{n_m-1}), we get a deformation of f_m into a map

$$f_m' : (D^{n_m}, S^{n_m-1}) \to (N', N).$$

But and N' are of the same homotopy type. Thus we get a deformation,

$$(\tilde{\phi}_t)_m : \tilde{N}_m \to N',$$

of $\tilde{N}_m$ onto N which extends $(\phi_t)_{m-1}$. Then $(\tilde{\phi}_t)_m$ can be extended uniquely to a two-sided invariant deformation retraction,

$$(\phi_t)_m : N_m \to N,$$

of N_m onto N as desired.

REMARK (7.2). A similar argument proves that, if N and N' are right (left) A-complexes and γ is a right (left) A-imbedding, then one can find a right (left) invariant deformation retraction ϕ_t of N' onto N.

8. **Proof of Theorem (6.3).** Suppose that M_1 is a two-sided A-complex which admits a Q_n-structure defined by the set of A-relations

$$\mathscr{M}_n(M_1) = (M_0, \cdots, M_n; \mu_{p,q}, A).$$

Denote the subset of $\mathscr{M}_n(M_1)$ in $k-1$ variables by

$$\mathscr{M}_{k-1}(\bar{M}_1) = (\bar{M}_0, \cdots, \bar{M}_{k-1}; \bar{\mu}_{p,q}, A).$$

Suppose ${}^{k-1}M$ is the RPT-complex generated by $\bar{M}_1$ modulo $\mathscr{M}_{k-1}(\bar{M}_1)$, and consider the complex

$$(8.1)_k \qquad {}^kN = {}^{k-1}M \cup M_k \times_A {}^{k-1}M,$$

where the complex $M_k \times_A {}^{k-1}M$ is attached to ${}^{k-1}M$ by the map

$$\bar{M}_k \times_A {}^{k-1}M \to {}^{k-1}M$$

which is induced by the multiplication in ${}^{k-1}M$. (Here $\bar{M}_k = {}^{k-1}M \cap M_k$, the subcomplex of ${}^{k-1}M$ of index k.) Now let ${}^{k-1}M$ act on ${}^kN$ on the right, thereby making ${}^kN$ a right ${}^{k-1}M$-complex. Hence, by Theorem (2.2) of §2, the sequence

$$(8.2)_k \qquad {}^{k-1}M \to {}^kN \to {}^kN//{}^{k-1}M,$$

is a quasi-fibration and, hence, a fiber sequence. The fibrations of the Hopf type are induced in the following way by (8.2). Denote by B_{k-1} the base-space of the universal quasi-fibration of ${}^{k-1}M$ (see (4.5) of §4). Then the sequence

$$(8.3)_k \qquad {}^kN \to {}^kN//{}^{k-1}M \xrightarrow{q_k} B_{k-1}$$

is a fiber sequence, where q_k is the classifying map of $(8.2)_k$. Observe that $(8.3)_k$ maps into $(8.3)_{k+1}$ in a natural fashion. Thus we get the commutative ladder,

$$\begin{array}{ccccccc}
{}^1N & \longrightarrow & {}^2N & \longrightarrow & \cdots & \longrightarrow & {}^nN \\
\downarrow & & \downarrow & & & & \downarrow \\
{}^1N//{}^0M & \longrightarrow & {}^2N//{}^1M & \longrightarrow & \cdots & \longrightarrow & {}^nN//{}^{n-1}M \\
\downarrow & & \downarrow & & & & \downarrow \\
B & \longrightarrow & B_1 & \longrightarrow & \cdots & \longrightarrow & B_{n-1},
\end{array}$$

where ${}^0M = A$, the vertical sequences are fiber sequences, and the maps of the lowest sequence are those induced by the homomorphisms ${}^kM \to {}^{k+1}M$. Thus, to finish the proof of the theorem, we need prove the following two lemmas only.

LEMMA (8.4). *The natural imbedding*

$$M_k \to {}^kN,$$

where M_k is the kth *complex of $\mathscr{M}_k(M_1)$ and ${}^kN$ is the complex defined above, is a homotopy equivalence.*

LEMMA (8.5). *The complexes*

$${}^kN//{}^{k-1}M \text{ and } (*)^k_A(M_1//A)$$

are of the same homotopy type.

Proof of Lemma (8.4). Recall that the RPT-complex ${}^{k-1}M$ is filtered by an ascending sequence of two-sided A-complexes

$$({}^{k-1}M)_r = \bigcup_{p=1}^{k-1} [\bar{M}_p, \bar{M}_{r-p}]_A$$

such that

$$[\bar{M}_p, \bar{M}_{r-p}]_A \cap \bigcup_{q<p} [\bar{M}_q, \bar{M}_{r-q}]_A = \bigcup_{q<p} [\bar{M}_{p-q}, \bar{M}_q, \bar{M}_{r-p}]_A$$

where $\bar{M}_i$ is the subcomplex of ${}^{k-1}M$ of index i, $i \geqq 1$ (see Corollary (3.7) of §3). It can be easily checked that this implies that

$$({}^{k-1}M)_r = [\bar{M}_k, \bar{M}_{r-k}]_A \cup \bigcup_{p=1}^{k-2} [\bar{M}_{k-p}, \bar{M}_{r-k+p}]_A,$$

where

$$[\bar{M}_{k-q}, \bar{M}_{r-k+q}]_A \cap \bigcup_{p=0}^{q-1} [\bar{M}_{k-p}, \bar{M}_{r-k+p}]_A = \bigcup_{p=0}^{q-1} [\bar{M}_{k-q}, \bar{M}_{q-p}, \bar{M}_{r-k+p}]_A .$$

Therefore we can filter the complex ${}^kN$ by an ascending sequence of two-sided A-subcomplexes,

$$({}^kN)_r = [M_k, \bar{M}_{r-k}]_A \cup \bigcup_{p=1}^{k-2} [\bar{M}_{k-p}, \bar{M}_{r-k+p}]_A, \quad r \geqq k,$$

such that

$$\begin{aligned}([M_k, \bar{M}_{r-k}]_A \cup \bigcup_{p=1}^{q-1} [\bar{M}_{k-p}, \bar{M}_{r-k+p}]_A) \cap [\bar{M}_{k-q}, \bar{M}_{r-k-q}]_A \\ = \bigcup_{p=0}^{q-1} [\bar{M}_{k-q}, \bar{M}_{q-p}, \bar{M}_{r-k+p}]_A\end{aligned}$$

where $\bar{M}_i$ is the subcomplex of ${}^{k-1}M$ of index i. Observe now that $\bar{M}_{k-1}$ is a two-sided A-deformation retract of M_k, and, for $p \geqq 1$, $\bar{M}_{k-p-1}$ is a two-sided A-deformation retract of $\bar{M}_{k-p}$. Hence $({}^kN)_r$ is made up of complexes each of which is of the homotopy type of $({}^kN)_{r-1}$. One can show inductively, using the Mayer-Vietoris sequences of the appropriate proper triads which correspond to the decomposition of $({}^kN)_r$ given above, that in constructing $({}^kN)_{r-1}$ one attaches two complexes of the homotopy type of $({}^kN)_{r-1}$ along a subcomplex of the homotopy type of $({}^kN)_{r-1}$. This establishes the lemma.

Proof of Lemma (8.5). If Q is a subcomplex of the special complex P, then denote by P/Q the special complex obtained from P by identifying Q with the base point. Observe that

$$(M_k//A)/(\bar{M}_k//A) = {}^kN//{}^{k-1}M .$$

Hence it suffices to prove that

$$(8.6)_k \qquad (M_k//A)/(\bar{M}_k//A) \text{ and } S^{k-1} \# (\#)_A^k(\bar{M}_1//A)$$

are of the same homotopy type.

Proceed by induction. If $k = 1$, then the assertion is trivially true. Suppose we have been able to prove $(8.6)_{m-1}$. To prove $(8.6)_m$, write

$$[p,q,r] = ([\bar{M}_p, \bar{M}_q, \bar{M}_r]_A)//A;$$

similarly, write $[p]$ for $(\bar{M}_p)//A$, and $[p,q]$ for $([\bar{M}_p, \bar{M}_q])//A$. Let us also put

$$[p] \#_A [q] \#_A [r] = [p,q,r]/[(p-1,q,r] \cup [p,q-1,r] \cup [p,q,r-1]).$$

Observe now that Corollary (3.7) of §3 implies that

$$[m]/[m-1] = \bigcup_{p=1}^{m-1} [m-p] \#_A [p]$$

where

$$[m-q] \#_A [q] \cap \bigcup_{p<q} [m-p] \#_A [p] = \bigcup_{p<q} [m-q] \#_A [q-p] \#_A [p].$$

To finish the proof we need the following lemma.

LEMMA $(8.7)_m$. *The complex*

$$\bigcup_{p=1}^{m-2} [m-p] \#_A [p]$$

is contractible,

Proof. Proceed by induction on m. If $m = 3$, then the Lemma is true obviously. Suppose $(8.7)_{s-1}$ is true, with $s \leqq m$. To prove $(8.7)_s$, argue by induction. Note that $[s-1] \#_A [1]$ is contractible, since M_{s-2} is a two-sided A-deformation retract of M_{s-1}. Suppose that we have been able to show that

$$\bigcup_{p=1}^{r-1} [s-p] \#_A [p], \quad r-1 < s-2,$$

is contractible. Observe that

$$\bigcup_{p=1}^{r} [s-p] \#_A [p] = [s-r] \#_A [r] \cup \bigcup_{p=1}^{r-1} [s-p] \#_A [p],$$

and

$$\bigcup_{p=1}^{r-1} [s-r] \#_A [r-p] \#_A [p] = [s-r] \#_A [r] \cap \bigcup_{p=1}^{r-1} [s-p] \#_A [p].$$

Next note that $[s-r] \#_A [r]$ is contractible, since M_{r-1} is a two-sided A-deformation retract of M_r, for $r = 2, \cdots, m \leqq k$. The complexes

$$\bigcup_{p=1}^{r-1} [s-p] \#_A [p] \text{ and } \bigcup_{p=1}^{r-1} [s-r] \#_A [r-p] \#_A [p]$$

are also contractible, the first by induction and the second because M_{r-1} is a two-sided deformation retract of M_r, for $2 \leqq r \leqq m$. Hence

$$\bigcup_{p=1}^{r} [s-p] \#_A [p], \qquad r \leqq s-2,$$

is contractible. This establishes the induction step, and hence, Lemma (8.7).

Now we can go back to the proof of $(8.6)_m$. Note that

$$\bigcup_{p=1}^{m-1} [m-p] \#_A [p] = \bigcup_{p=1}^{m-2} [m-p] \#_A [p] \cup [1] \#_A [m-1],$$

and

$$\begin{aligned} \bigcup_{p=1}^{m-2} [m-p] \#_A [p] \cap [1] \#_A [m-1] &= \bigcup_{p=1}^{m-2} [1] \#_A [m-p-1] \#_A [p] \\ &= \bigcup_{q=1}^{m-2} [1] \#_A [m-1-q] \#_A [q]. \end{aligned}$$

The induction hypothesis implies that

$$\begin{aligned} \bigcup_{q=1}^{m-2} [m-1-q] \#_A [q] &= ([m-1])/[m-2] \\ &\sim S^{m-2} \# (\#)_A^{m-1} [1]. \end{aligned}$$

This proves $(8.6)_m$ and, hence, the lemma.

9. **Proof of Theorem (5.7).** Suppose B is a simply connected special complex and F, a connected subcomplex. Assume that A and U are the RPT-complex and right A-complex representing $\Omega(B)$ and $\Omega(B, F)$, as in (5.1) and (5.2) of §5. Recall that Condition (5.5) of §5 says that there is a two-sided A-complex M_1 and a right invariant map,

$$\phi : U \to M_1,$$

which is a homotopy equivalence. Suppose that

$$\mathscr{M}_n(M_1) = (M_0, \cdots, M_n; \mu_{p,q}, A)$$

is a set of A-relations which defines a Q_n-structure on M_1 (see Definition (5.4)), and let

$$B \xrightarrow{\eta_1} B_1 \xrightarrow{\eta_2} \cdots \xrightarrow{\eta} B_n$$

be the ascending sequence of complexes defined by the sequence of RPT-complexes

$$A \to {}^1M \to \cdots \to {}^nM,$$

where ${}^kM$ is the RPT-complex generated by M_1 modulo $\mathscr{M}_k(M_1)$ and B_k is the base-space of the universal quasi-fibration of ${}^kM$ (see §4). As in §5, let

$$F_k = \{\alpha \in \tilde{P}(B_k) \,|\, \alpha(0) \in B\},$$

where $\tilde{P}(B_k)$ is the space of Moore paths ending at the 0-cell b^0. Observe that $\Omega(B_k)$ acts on the left on F_k. The first stage of the proof is just the replacement of F_k by an equivalent left A-complex. Now note that, according to §4, one can find a left universal quasi-fibration,

$$p'_k : E'({}^kM) \to B_k,$$

such that $E'({}^jM)$ is a subcomplex of $E'({}^kM)$, the imbedding being a map of a left ${}^jM$-complex into a left ${}^kM$-complex, $k > j$ (${}^0M = A$, by definition). Moreover, there is a homomorphism u_k of ${}^kM$ into $\Omega(B_k)$ which extends to a map,

$$u'_k : E'({}^kM) \to \tilde{P}(P_k),$$

of a left ${}^kM$-complex into a left $\Omega(B_k)$-space such that

$$\begin{array}{ccccc}
{}^kM & \longrightarrow & E'({}^kM) & \xrightarrow{p'_k} & B_k \\
\downarrow{\scriptstyle u_k} & & \downarrow{\scriptstyle u'_k} & & \downarrow{\scriptstyle =} \\
\Omega(B_k) & \longrightarrow & \tilde{P}(B_k) & \xrightarrow{q'_k} & B_k
\end{array}$$

is a commutative diagram. Here q'_k is the map that sends a path to its initial point. Observe that u'_k takes ${}^kM \times_A E'(A)$ into F_k and, hence, induces the following commutative diagram:

$$\begin{array}{ccccc}
{}^kM & \longrightarrow & {}^kM \times_A E'(A) & \longrightarrow & B \\
\downarrow{\scriptstyle u_k} & & \downarrow{\scriptstyle u'_k} & & \downarrow{\scriptstyle =} \\
\Omega(B_k) & \longrightarrow & F_k & \longrightarrow & B.
\end{array}$$

This implies the following assertion.

(9.1) *u'_k is a homotopy equivalence.*

Thus we can replace F_k by the left ${}^kM$-complex ${}^kM \times_A E'(A)$. Consider now the following diagram (see end of §4),

$$\begin{array}{ccccccc}
{}^kM//A & \xleftarrow{\approx\rho_k} & {}^kM \times_A E'(A) & \xrightarrow{d_k} & B & \xrightarrow{f_k} & B_k \\
\uparrow{\scriptstyle\bar\delta_k} & & \uparrow{\scriptstyle\tilde\delta_k} & & \uparrow{\scriptstyle =} & & \uparrow{\scriptstyle =} \\
M_k//A & \xleftarrow{\approx} & M_k \times_A E'(A) & \longrightarrow & B & \xrightarrow{f_k} & B_k \\
\uparrow{\scriptstyle\bar\Phi_k} & & \uparrow{\scriptstyle\tilde\phi_k} & & \uparrow{\scriptstyle =} & & \uparrow{\scriptstyle =} \\
F = U//A & \xleftarrow{\approx} & U \times_A E'(A) & \longrightarrow & B & \xrightarrow{f_k} & B_k \\
\downarrow{\scriptstyle i} & & \downarrow & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} \\
B & \xleftarrow{\approx} & E(A) \times_A E'(A) & \longrightarrow & B & \xrightarrow{f_k} & B_k,
\end{array}$$

where the horizontal maps to the left of the B column are those induced by the projections on the corresponding factors, while the rest of the horizontal maps and all the vertical maps are those induced by the natural imbeddings, (see end of §4). Note that the sequence

$$ {}^kM \times_A E'(A) \xrightarrow{d_k} B \xrightarrow{f_k} B_k $$

is a fiber sequence which is equivalent to the fiber sequence

$$ \Omega(B_k, B) = F_k \xrightarrow{i_k} B \longrightarrow B_k, $$

i_k being the map that sends a path to its initial point. Observe also that the horizontal maps pointing to the left are all quasi-fibrations with contractible fiber $E'(A)$ and hence are homotopy equivalences. Let ρ_k^{-1} be a homotopy-inverse for ρ_k, and set, by definition,

$$ \gamma_k = \rho_k^{-1}\bar\delta_k\bar\Phi_k. $$

Then γ_k takes $F = U//A$ into $F_k = {}^kM \times_A E'(A)$, and

$$ \lambda_k\gamma_k \sim \gamma_{k+1} \text{ and } gi = d_k\gamma_k = i_k\gamma_k $$

(note that d_k and i_k become the same when ${}^kM \times_A E'(A)$ and F_k are identified), where g is a homotopy equivalence of B and λ_k is the imbedding of ${}^kM \times_A E'(A)$ into ${}^{k+1}M \times_A E'(A)$ induced by the imbedding of ${}^kM$ into ${}^{k+1}M$ (or, equivalently, the imbedding of F_k into F_{k+1} induced by the imbedding f_k of B into B_k).

To finish the proof of the theorem, consider the injection

$$ \delta_k : M_k \to {}^kM. $$

Denote by V the special complex $S^{k-1}\#(\#)_A^{k+1}(M_1//A)$ (see §6). Then the adjoint $\bar p_{k+1}$ of the fibration

$$ p_{k+1} : (*)_A^{k+1}(M_1//A) \to B_k $$

of Theorem (6.3) induces a homomorphism

$$(\bar{p}_{k+1})_{\infty}\colon V_{\infty} \to {}^{k}M,$$

where V_{∞} is the reduced product complex of James. In the usual fashion, Theorem (6.3) implies that the map

$$v\colon V_{\infty} \times M_k \to {}^{k}M \tag{9.2}$$

which takes the point (x, y) to $(\bar{p}_{k+1})_{\infty}(x)\delta_k(y)$ is a homotopy equivalence. Consider now $V_{\infty} \times M_k$ a right A-complex by allowing A to act on the right factor. Observe that v becomes a right invariant map, and, hence, by Remark (7.2), there is a right A-map,

$$r\colon {}^{k}M \to V_{\infty} \times M_k,$$

such that $vr \sim 1$ and $rv \sim 1$. Define

$$\tilde{\beta}_k\colon {}^{k}M \to M_k$$

to be the composition of r and the projection on the second factor. It is a right map and, hence, induces a map

$$\beta_k\colon {}^{k}M // A \to M_k // A$$

such that $\beta_k \gamma_k \sim 1$.

This finishes the proof of part (a) of the theorem. Part (b) can be established as follows.

If $F \sim M_1 // A$ is $(c-1)$-connected, then one can replace in a straight forward fashion M_1 by a two-sided A-complex $\tilde{M}_1$ of the same homotopy type and such that the cells of lowest dimension in $M_1 // A$ are of dimension c. Hence $(*)_A^{k+1}(M_1 // A)$ is of the homotopy type of $(*)_A^{k+1}(\tilde{M}_1 // A)$ (see Lemma (7.1) of §7). But the cells of lowest dimension in $(*)_A^{k+1}(\tilde{M}_1 // A)$ have dimension $(k+1)c + k$. Hence $(*)_A^{k+1}(M_1 // A)$ is $((k+1)c + (k-1))$-connected, and V is $((k+1)c + (k-2))$-connected. Therefore, the injection

$$\delta_k\colon M_k \to {}^{k}M$$

induces an isomorphism

$$(\delta_k)_*\colon \pi_j(M_j) \to \pi_j({}^{k}M),$$

if $j \leqq (k+1)c + (k-2)$.

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